

# The Wiener dimension of a graph

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## Abstract

The Wiener dimension of a connected graph is introduced as the number of different distances of its vertices. For any integer  $d$  and any integer  $k$ , a graph of diameter  $d$  and of Wiener dimension  $k$  is constructed. An infinite family of non-vertex-transitive graphs with Wiener dimension 1 is also presented and it is proved that a graph of dimension 1 is 2-connected.

**Keywords:** Wiener index; Cartesian product graphs; vertex-transitive graphs

**AMS subject classification (2010):** 05C12, 05C25, 92E10

## 1 Introduction

The distance considered in this paper is the usual shortest path distance. We assume throughout the paper that all graphs are connected unless stated otherwise. For terms not defined here we refer to [13].

Let  $G$  be a graph and  $u \in V(G)$ . Then the *distance* of  $u$  is

$$d_G(u) = \sum_{v \in V(G)} d_G(u, v).$$

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In location theory, sets of vertices with the minimum (or maximum) distance in a graph play a special role because they form target sets for locations of facilities. The set of vertices of a graph  $G$  that minimizes the distance is called the median set of  $G$ . The framework can be made more general (at least) in two ways: by considering the sum of distances to a specified multiset of vertices (such multisets are referred to as profiles) and by considering weighted graphs. For more information in this direction of research see [1, 2, 9, 14].

The famous *Wiener index* of a graph  $G$  is defined as

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} d_G(u).$$

Suppose now that  $\{d_G(u) \mid u \in V(G)\} = \{d_1, d_2, \dots, d_k\}$ . Assume in addition that  $G$  contains  $t_i$  vertices of distance  $d_i$ ,  $1 \leq i \leq k$ . Then the Wiener index of  $G$  can be expressed as

$$W(G) = \frac{1}{2} \sum_{i=1}^k t_i d_i. \tag{1}$$

We therefore say that the *Wiener dimension*  $\dim_W(G)$  of  $G$  is  $k$ . That is, in this paper we introduce the Wiener dimension of a graph as the number of different distances of its vertices.

The paper is organized as follows. In the next section the Wiener dimension is given for some classes of graphs. Then, in Section 3, we construct for any integer  $d$  and any integer  $k$  a graph of diameter  $d$  and of Wiener dimension  $k$ . In the final section we consider graphs of Wiener dimension 1. An infinite family of non-vertex-transitive graphs with Wiener dimension 1 is constructed and it is proved that a graph of dimension 1 is 2-connected.

## 2 Some examples

It is easy to see that  $\dim_W(K_n) = \dim_W(C_n) = \dim_W(P) = 1$ , where  $P$  is the Petersen graph, the intrinsic reason being that all these graphs are vertex-transitive, cf. Section 4. It is also clear that  $\dim_W(K_{n,m}) = 2$  as soon as  $n \neq m$ , and it is not difficult to infer that  $\dim_W(P_n) = \lceil n/2 \rceil$  for any  $n \geq 1$ . For a sporadic example of a graph  $G$  with  $\dim_W(G) = 4$  see Fig. 1, where the graph  $G$  is shown together with the distances of its vertices.

For a slightly more elaborate example consider the family of cyclic phenylenes, a class of graphs arising in mathematical chemistry [3, 16]. These graphs are composed of cyclically attached hexagons and squares, as shown in Fig. 2 for the cyclic phenylene  $R_5$ . The definition of  $R_k$ ,  $k \geq 3$ , should be clear from this example.

Let  $u, v, w$  be vertices of  $R_h$ , where  $u$  is a vertex of the inner long cycle,  $v$  a vertex of degree 3 on the outer long cycle, and  $w$  a vertex of degree 2. Then it is not difficult to see that for any  $k \geq 3$ ,  $d_{R_k}(u) = 3k^2 + 6k$ ,  $d_{R_k}(v) = 3k^2 + 12k - 12$ ,

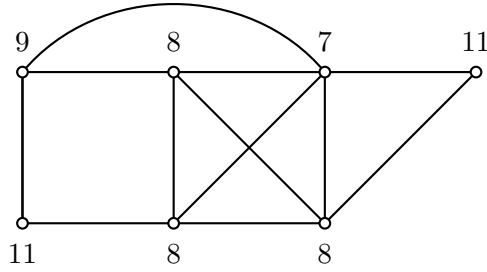


Figure 1: An asymmetric graph  $G$  with  $\dim_W(G) = 4$

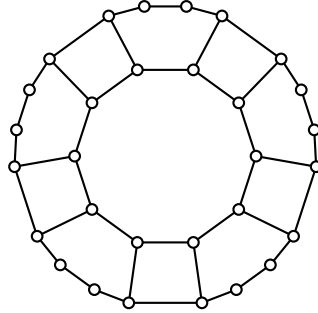


Figure 2: The cyclic phenylene  $R_5$

and  $d_{R_k}(w) = 3k^2 + 18k - 24$ . Since  $d_{R_k}(u) < d_{R_k}(v) < d_{R_k}(w)$  holds for  $k \geq 3$ , we have  $\dim_W(R_k) = 3$ ,  $k \geq 3$ .

Using (1), we thus also get:

$$\begin{aligned} W(R_k) &= \frac{1}{2}2k((k^2 + 6k) + (3k^2 + 12k - 12) + (3k^2 + 18k - 24)) \\ &= 9k^3 + 36k^2 - 36k. \end{aligned}$$

### 3 Graphs with given diameter and Wiener dimension

The *Inverse Wiener index problem* is to find a graph from a certain class of graphs with a given value of the Wiener index. The inverse Wiener index problem was solved for general graphs by Goldman et al. [7]: for every positive integer  $n$  except 2 and 5 there exists a graph  $G$  such that the Wiener index of  $G$  is  $n$ . Lepovič and Gutman [8] conjectured that all but 49 positive integers are Wiener indices of trees. The conjecture was independently proved in [10] and [11]. For some recent developments on the inverse Wiener problem see [12, 15].

In this section we start the Inverse Wiener dimension problem. As the first result

in this direction we prove that for any integer  $d$  and any integer  $k$  there exists a graph of diameter  $d$  and of Wiener dimension  $k$ . The corresponding construction uses two main ingredients, graphs of diameter 2 and Cartesian products of graphs. We begin with the following

**Lemma 3.1** *Let  $G$  be a graph of order  $n$ , diameter 2, and let  $v \in V(G)$ . Then  $d_G(v) = 2n - \deg(v) - 2$ . In particular,  $\dim_W(G) = |\{\deg(u) \mid u \in V(G)\}|$ .*

**Proof.** Since  $\text{diam}(G) = 2$ , we have  $d_G(v) = \deg(v) + 2(n - \deg(v) - 1) = 2n - \deg(v) - 2$ . Therefore,  $d_G(u) = d_G(w)$  if and only if  $\deg(u) = \deg(w)$ .  $\square$

Recall that the Cartesian product  $G \square H$  of graphs  $G$  and  $H$  has vertex set  $V(G \square H) = V(G) \times V(H)$  and  $(g, h)$  is adjacent to  $(g', h')$  if  $g = g'$  and  $hh' \in E(H)$ , or  $h = h'$  and  $gg' \in E(G)$ . This graph operation is associative, hence we may consider powers of graphs with respect to it. Powers of  $K_2$  are known as hypercubes, the  $d$ -tuple power is denoted with  $Q_d$ .

Let  $(g, h), (g', h') \in V(G \square H)$ , then it well-known (cf. [6, Proposition 5.1]) that  $d_{G \square H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h')$ . Note that this fact in particular implies that  $\text{diam}(G \square H) = \text{diam}(G) + \text{diam}(H)$ . Moreover,

$$\begin{aligned}
d_{G \square H}((g, h)) &= \sum_{(g', h') \in V(G \square H)} d_{G \square H}((g, h), (g', h')) \\
&= \sum_{(g', h') \in V(G \square H)} (d_G(g, g') + d_H(h, h')) \\
&= |V(H)| \sum_{g' \in V(G)} d_G(g, g') + |V(G)| \sum_{h' \in V(H)} d_H(h, h') \\
&= |V(H)|d_G(g) + |V(G)|d_H(h). \tag{2}
\end{aligned}$$

Equation (2) has several consequences. First of all,

$$\dim_W(G \square H) = |\{|V(H)|d_G(g) + |V(G)|d_H(h) \mid g \in V(G), h \in V(H)\}|.$$

Consequently,

$$\max \{\dim_W(G), \dim_W(H)\} \leq \dim_W(G \square H) \leq \dim_W(G)\dim_W(H).$$

For our purposes, we apply Equation (2) as follows:

**Corollary 3.2** *Let  $G$  be a graph and let  $H$  be a graph with  $\dim_W(H) = 1$ . Then  $\dim_W(G \square H) = \dim_W(G)$ .*

**Proof.** Since  $\dim_W(H) = 1$ , there exists a constant  $s$  such that  $d_H(h) = s$  for any  $h \in V(H)$ . Then by (2),  $d_{G \square H}((g, h)) = |V(H)|d_G(g) + |V(G)|s$  for any vertex  $(g, h) \in V(G \square H)$ . The conclusion is then clear.  $\square$

Now everything is ready for the main result of this section. Since the only graphs of diameter one are complete graphs, their Wiener dimension is not interesting. However, for diameter at least two we have:

**Theorem 3.3** *For any  $d \geq 2$  and any  $k \geq 1$  there exists a graph  $G$  such that  $\text{diam}(G) = d$  and  $\text{dim}_W(G) = k$ .*

**Proof.** For  $d \geq 2$  and  $k = 1$  note that  $\text{diam}(C_{2d}) = d$  and  $\text{dim}_W(C_{2d}) = 1$ . Hence it remains to prove the theorem for any  $d \geq 2$  and any  $k \geq 2$ .

Let  $G_k$ ,  $k \geq 2$ , be the graph defined as follows:  $V(G_k) = \{1, 2, \dots, k+1\}$  and  $ij \in E(G_k)$  whenever  $i + j \leq k + 2$ . The vertex 1 of  $G_k$  is of degree  $k$  and since  $G_k$  is not a complete graph we first get that  $\text{diam}(G_k) = 2$ . Moreover, the degree sequence of  $G_k$  is  $k, k-1, \dots, k/2+1, k/2, k/2, k/2-1, \dots, 2, 1$  when  $k$  is even and  $k, k-1, \dots, (k+1)/2+1, (k+1)/2, (k+1)/2, (k+1)/2-1, \dots, 2, 1$  when  $k$  is odd. In any case, Lemma 3.1 implies that for any  $k \geq 2$ ,  $\text{dim}_W(G_k) = k$ .

So  $G_k$ ,  $k \geq 2$ , is a graph with  $\text{diam}(G_k) = 2$  and  $\text{dim}_W(G_k) = k$ . Let now  $d \geq 3$  and let  $H$  be an arbitrary vertex-transitive graph of diameter  $d-2$ . (Say,  $H = Q_{d-2}$ .) Then since  $\text{diam}(G \square H) = \text{diam}(G) + \text{diam}(H)$ , we have  $\text{diam}(G_k \square H) = 2 + (d-2) = d$  and by Corollary 3.2,  $\text{dim}_W(G_k \square H) = k$ .  $\square$

## 4 On graphs with Wiener dimension 1

In this section we consider graphs with Wiener dimension 1. Examples of such graphs are all vertex-transitive graphs, but there are other examples as well. We construct an infinite family of non-vertex-transitive graphs with Wiener dimension 1. We also show that a graph of dimension 1 is 2-connected.

Let  $u, v \in V(G)$  and  $\alpha \in \text{Aut}(G)$  such that  $\alpha(u) = v$ . Then, since  $\alpha$  preserves distances, we have

$$d_G(u) = \sum_{v \in V(G)} d_G(u, v) = \sum_{v \in V(G)} d_G(\alpha(u), \alpha(v)) = \sum_{v \in V(G)} d_G(v, \alpha(v)) = d_G(v).$$

This means that vertices of the same orbit of the automorphism group of a graph  $G$  have the same distance; in particular,  $\text{dim}_W(G) = 1$  holds for vertex-transitive graphs  $G$ . Using (1) again we thus have:

**Corollary 4.1** ([17]) *Let  $G$  be a vertex-transitive graph and  $u \in V(G)$ . Then  $W(G) = |V(G)|d_G(u)/2$ .*

Zhang and Li [17] extended this result by considering a subgroup of  $\text{Aut}(G)$  and its orbits. The approach using Corollary 4.1 (and its edge-transitivity variation) was recently applied in [5].

On the other hand,  $d_G(u) = d_G(v)$  does not necessarily imply that  $u$  and  $v$  are in the same orbit of  $\text{Aut}(G)$  as we have already observed in Fig. 1. Note in addition that the graph  $G$  from the figure is asymmetric.

**Theorem 4.2** *There exists a family of graph  $\{G_k\}_{k \geq 0}$  such that  $G_k$  is non-vertex-transitive and  $\text{dim}_W(G_k) = 1$ .*

**Proof.** Set  $G_0$  to be the Tutte 12-cage. It is well-known that it is not vertex-transitive and we checked (using computer) that  $\dim_{\mathbb{W}}(G_0) = 1$ . Set now  $G_k = G_0 \square Q_k$ ,  $k \geq 1$ , where  $Q_k$  is the  $k$ -dimensional cube. Since a Cartesian product has transitive automorphism group if and only if every factor has transitive automorphism group, cf. [6, Theorem 6.17], we get that  $G_k$ ,  $k \geq 1$ , is not vertex-transitive. On the other hand, since  $Q_k$  is vertex-transitive,  $\dim_{\mathbb{W}}(Q_k) = 1$  and hence by Corollary 3.2,  $\dim_{\mathbb{W}}(G_k) = 1$ .  $\square$

It is clear from the above proof that the same effect could also be obtained by considering Cartesian powers of the Tutte 12-cage, that is,  $G_0^k$ ,  $k \geq 1$ , are not vertex-transitive but have Wiener dimension 1.

The Tutte 12-cage is an example of a semisymmetric graphs, where a graph  $G$  is called *semisymmetric* if  $G$  is regular, edge-transitive but not vertex-transitive [4]. Of course,  $1 \leq \dim_{\mathbb{W}}(G) \leq 2$  holds for any semisymmetric graph  $G$ . We have checked the Wiener dimension of the four smallest cubic semisymmetric graphs. Interestingly, two of them, namely the Gray graph and the Ljubljana graph have Wiener dimension 2, while the other two—the 110-Iofinova-Ivanov graph and the Tutte 12-cage—have Wiener dimension 1. It seems an interesting problem to characterize semisymmetric graphs with Wiener dimension 2.

We conclude with the following structural property of graphs of dimension 1:

**Proposition 4.3** *If  $\dim_{\mathbb{W}}(G) = 1$ , then  $G$  is 2-connected.*

**Proof.** Suppose on the contrary that  $x$  is a cut-vertex of  $G$ . Let  $G_1$  be an arbitrary connected component of  $G - x$  and let  $G_2$  be the remaining graph of  $G - x$ . Let  $|G_1| = n_1$  and  $|G_2| = n_2$ , so that  $|G| = n_1 + n_2 + 1$ . Let  $y$  be a neighbor of  $x$  from  $G_1$ . Then

$$\begin{aligned} d_{G_1}(x) &\leq d_{G_1}(y) + n_1, \\ d_{G_2}(y) &= d_{G_2}(x) + n_2. \end{aligned}$$

Summing up these inequalities we get

$$d_{G_1}(x) + d_{G_2}(x) + n_2 \leq d_{G_1}(y) + d_{G_2}(y) + n_1.$$

Since  $d_G(x) = d_{G_1}(x) + d_{G_2}(x)$  and  $d_G(y) = d_{G_1}(y) + d_{G_2}(y) + 1$  it follows that

$$d_G(x) + n_2 \leq (d_G(y) - 1) + n_1.$$

Therefore,  $n_2 < n_1$ . On the other hand, selecting a neighbor of  $x$  in  $G_2$ , an analogous argument gives  $n_1 < n_2$ , a contradiction.  $\square$

Besides the cycles, we know of no graph of Wiener dimension 1 that is not 3-connected.

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