Hamming dimension of a graph - the case of Sierpiński graphs

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Abstract

The Hamming dimension of a graph G is introduced as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph. An upper bound is proved for the Hamming dimension of Sierpiński graphs S_k^n, $k \geq 3$. The Hamming dimension of S_3^n grows as 3^{n-3}. Several explicit embeddings are constructed along the way, in particular into products of generalized Sierpiński triangle graphs. The canonical isometric representation of Sierpiński graphs is also explicitly described.

Keywords: Hamming graphs; Hamming dimension; Sierpiński graphs; Cartesian product of graphs; induced embeddings; isometric embeddings

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1 Introduction

Several graph dimensions based on embeddings into product graphs have been studied by now. The *isometric dimension* of G is the largest number of factors of a Cartesian product graph, such that G is an irredundant, isometric subgraph of the product [6]. The *strong isometric dimension* is defined analogously, except that one embeds into the strong product of paths [3, 4], while the *lattice dimension* is defined via embeddings into Cartesian products of paths [2, 13]. The lattice dimension of a graph G is finite if and only if G is isometrically embeddable into some hypercube. For additional related dimensions see [8, Section 15.3].

In this paper we introduce the *Hamming dimension* $\mathrm{Hdim}(G)$ of a graph G as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph. Clearly, $\mathrm{Hdim}(G) = 1$ if and only if G is a complete graph. Moreover, $\mathrm{Hdim}(G) < \infty$ if and only if G admits a certain edge labeling, see Theorem 3.1 below. Note that $K_4 - e$ is the smallest graph with $\mathrm{Hdim}(K_4 - e) = \infty$. The general problem of determining the Hamming dimension of a graph seems very demanding, here we will study this concept on Sierpiński graphs.

Sierpiński graphs S_k^n were introduced and studied for the first time in [15]. The study was motivated in part by the fact that for $k = 3$ these graphs are isomorphic to the Tower of Hanoi graphs [9] and in part by topological studies. For details about the latter motivation see Lipscomb's book [20]. Sierpiński graphs were later independently introduced in [23].

The graphs S_k^n were investigated from numerous points of view, we recall some of them. These graphs contain (essentially) unique 1-perfect codes [16], a classification of their covering codes is given in [5]. In [7] a shorter proof is given for the uniqueness of 1-perfect codes and their optimal $L(2,1)$-labelings are presented. Equitable $L(2,1)$-labelings were later studied in [1]. The crossing number of Sierpiński graphs and their natural regularizations was studied in [17], giving first infinite families of graphs of fractal nature for which the crossing number was determined (up to the crossing number of complete graphs). Metric properties of Sierpiński graphs were investigated in [12, 21]. To determine the chromatic number of these graphs is easy, while in [11] it is proved that they are in edge- and total coloring class 1, except those isomorphic to a complete graph of odd or even order, respectively. Recently, the hub number of Sierpiński graphs was determined in [19].

As already said, Sierpiński graphs are closely related to the Tower of Hanoi. In [24], Romik used the Sierpiński labeling of S_3^n to construct an appealing finite automaton that solves the decision problem of whether the largest disc moves once or twice on a shortest path from a regular to another regular configuration in the Tower of Hanoi problem. For connections between the Sierpiński graphs S_3^n (alias Hanoi graphs) and the Stern's diatomic sequence see [10].

We proceed as follows. In the rest of this section we give necessary definitions. In the next section Sierpiński graphs and generalized Sierpiński triangle graphs are introduced and some of their properties recalled. Then, in Section 3, the theory from [18] on induced embeddings into Hamming graphs and more generally, into Cartesian product graphs, is recalled. It is applied to describe induced embeddings of Sierpiński graphs into Cartesian products of generalized Sierpiński triangle graphs. In Section 4 it is proved that for any $n \geq 2$,

$$\mathrm{Hdim}(S_3^n) \geq \frac{7}{4} \cdot 3^{n-3} + 3 \cdot 2^{n-4} + \frac{3}{2}n - \frac{9}{4}\,.$$

In the subsequent section an upper bound for $\mathrm{Hdim}(S_k^n)$, $k \geq 3$. Together with the lower bound it implies that $\mathrm{Hdim}(S_3^n)$ asymptotically grows as 3^{n-3}. As proved in [6], an irredundant *isometric* embedding into the largest number of factors is unique and called the canonical isometric representation. In the last section we explicitly describe this embedding of S_k^n.

The *Cartesian product* $G \,\Box\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, where the vertex (g, h) is adjacent to the vertex (g', h') whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. The Cartesian product is commutative and associative, products whose all factors are complete are called *Hamming graphs*. The *dimension of a Hamming graph* is the number of its factors, that is, the number of coordinates of its vertices. We say that a graph G is an *irredundant subgraph* of $\Box_{i=1}^{p} G_i$ if each G_i has at least two vertices and any vertex of G_i appears as a coordinate of some vertex of G. With these concepts we can thus state:

$$\mathrm{Hdim}(G) = \max\{p \mid G \text{ is irredundant induced subgraph of } \Box_{i=1}^{p} K_{p_i}\}\,.$$

The *distance* $d(u, v) = d_G(u, v)$ between vertices u and v of a graph G is the length of a shortest u, v-path in G. A subgraph H of a graph G is *isometric* if for each pair of vertices u, v of H there exists a shortest u, v-path in G that lies entirely in H. Finally, by a *labeled graph* we mean a graph together with a labeling of its edges.

2 Sierpiński graphs

The *Sierpiński graph* S_k^n, $k, n \geq 1$, is defined on the vertex set $\{1, \ldots, k\}^n$, two different vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ being adjacent if and only if there exists an $h \in \{1, \ldots, n\}$ such that

(i) $u_t = v_t$, for $t = 1, \ldots, h - 1$;
(ii) $u_h \neq v_h$; and
(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, \ldots, n$.

In the rest we will use abbreviation $\langle u_1 u_2 \ldots u_n \rangle$ for $(u_1, u_2, \ldots, u_n)$. On figures, this will be further simplified to $u_1 u_2 \ldots u_n$. The Sierpiński graph S_3^4 together with the corresponding vertex labeling is shown on Fig. 1.

A vertex of the form $\langle ii \ldots i \rangle$ of S_k^n is called an *extreme vertex*. Note that S_k^n contains k extreme vertices and that $|V(S_k^n)| = k^n$. Let $n \geq 2$, then for $i = 1, \ldots, k$, let iS_k^{n-1} be

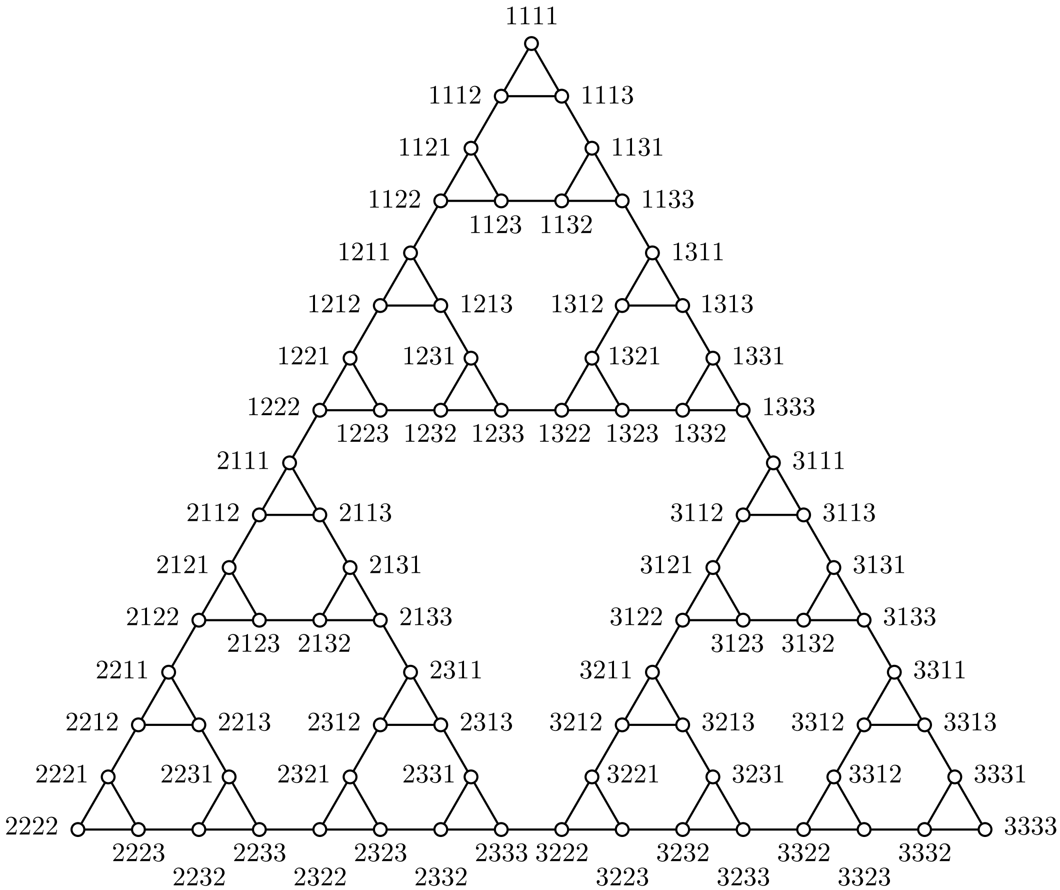

Figure 1: The Sierpiński graph S_3^4

the subgraph of S_k^n induced by the vertices of the form $\langle iv_2 \dots v_n \rangle$. More generally, for given $i_1, \dots, i_r \in \{1, \dots, k\}$, we denote with $i_1 \dots i_r S_k^{n-r}$ the subgraph of S_k^n induced by the vertices of the form $\langle i_1 \dots i_r v_{r+1} \dots v_n \rangle$. Note that iS_k^{n-1} is isomorphic to S_k^{n-1}, and, more generally, $i_1 \dots i_r S_k^{n-r}$ is isomorphic to S_k^{n-r}.

An edge of S_k^n of the form $\langle u_1 u_2 \dots u_{n-1} i \rangle \langle u_1 u_2 \dots u_{n-1} j \rangle$, $i \neq j$, will be called a *clique edge*. A clique edge is contained in a unique subgraph K_k of S_k^n. The other edges will be called *non-clique edges*. Let $i \neq j$. Then the edge $\langle ijj \dots j \rangle \langle jii \dots i \rangle$ is the unique edge between iS_k^{n-1} and jS_k^{n-1}. It will be denoted with $e_{ij}^{(n)} = e_{ji}^{(n)}$. Consider the subgraph $i_1 \dots i_r S_k^{n-r}$ of S_k^n. Then the edge between $\langle i_1 \dots i_r j \ell \dots \ell \rangle$ and $\langle i_1 \dots i_r \ell j \dots j \rangle$ will be denoted $i_1 \dots i_r e_{j\ell}^{(n-r)}$.

Setting

$$\rho_{i,j} = \begin{cases} 1 & i \neq j\,, \\ 0 & i = j\,, \end{cases}$$

the following holds:

Lemma 2.1 [15] *Let $\langle u_1 u_2 \dots u_n \rangle$ and $\langle ii \dots i \rangle$ be vertices of S_k^n. Then*

$$d_{S_k^n}(\langle u_1 u_2 \dots u_n \rangle, \langle ii \dots i \rangle) = \rho_{u_1,i}\rho_{u_2,i} \dots \rho_{u_n,i}\,,$$

where the right-hand side is a binary number with digits $\rho_{u_j,i}$. Moreover, a shortest path between $\langle u_1 u_2 \dots u_n \rangle$ and $\langle ii \dots i \rangle$ is unique.

The unique path in S_k^n between $\langle ii \dots i \rangle$ and $\langle jj \dots j \rangle$ will be denoted $P_{ij}^{(n)}$. Similarly, in the subgraph $i_1 \dots i_r S_k^{n-r}$, there is a unique path between $\langle i_1 \dots i_r jj \dots j \rangle$ and $\langle i_1 \dots i_r \ell\ell \dots \ell \rangle$, it will be denoted $i_1 \dots i_r P_{j\ell}^{(n-r)}$. By the uniqueness of the shortest paths between extreme vertices, it follows that there is also a unique shortest cycle of S_k^n containing the edges $e_{ij}^{(n)}$, $e_{j\ell}^{(n)}$, and $e_{\ell i}^{(n)}$, where $i, j, \ell \in \{1, 2, \dots, k\}$ are pairwise different. This cycle will be denoted $C_{ij\ell}^{(n)}$.

One of our embeddings will be an embedding into the Cartesian product of generalized Sierpiński triangle graphs, a class of graphs introduced in [14] as 2-parametric Sierpiński gasket graphs. For $n \geq 1$ and $k \geq 3$, the *generalized Sierpiński triangle graph* $\widehat{S_k^n}$ is the graph obtained from S_k^n by contracting all non-clique edges of S_k^n. Note that $\widehat{S_k^1} = K_k$ ($k \geq 3$). For $\widehat{S_4^2}$ see Fig. 2, where $\{i, j\}$ denotes the vertex obtained by contracting the edge $\langle ij \rangle \langle ji \rangle$.

3 Embeddings into products of generalized Sierpiński triangle graphs

In this section we first summarize the theory developed in [18] about induced embeddings of graphs into Hamming graphs.

Let G be a connected graph and let $\mathcal{F} = \{F_1, F_2, \dots, F_p\}$ be a partition of $E(G)$. Such a partition yields the corresponding labeling $\ell : E(G) \to \{1, 2, \dots, p\}$ by setting $\ell(e) = i$ for $e \in F_i$. For our purpose, the following conditions of a labeling are crucial:

Condition A. *Let G be a labeled graph. Then edges of a triangle have the same label.*

Condition B. *Let G be a labeled graph and let u and v be arbitrary vertices of G with $d_G(u, v) \geq 2$. Then there exist different labels i and j which both appear on any induced u, v-path.*

Now we can recall:

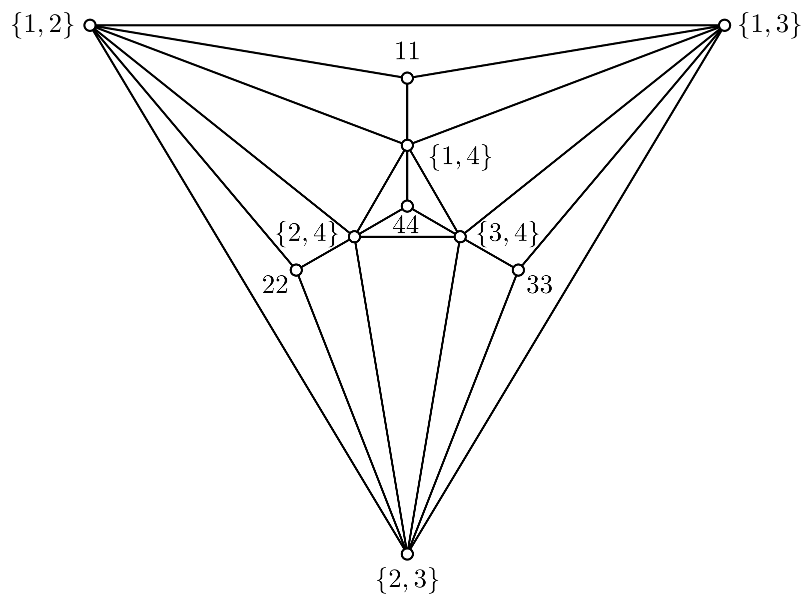

Figure 2: The generalized Sierpiński triangle graph $\widehat{S_4^2}$

Theorem 3.1 [18] *Let G be a connected graph. Then* $\mathrm{Hdim}(G) < \infty$ *if and only if there exists a labeling of G that fulfills Conditions A and B.*

The proof of Theorem 3.1 is constructive in the following way. If G is an induced subgraph of a Hamming graph with p factors, then the labeling of G that respects the projection of the edge uses p labels and satisfies Conditions A and B. Conversely, let $\mathcal{F} = \{F_1, \ldots, F_p\}$ be a partition of $E(G)$ such that the corresponding labeling ℓ fulfills Conditions A and B. For every $i = 1, \ldots, p$, define the graph G/F_i whose vertices are the connected components of $G \setminus F_i$, two components C and C' being adjacent in G/F_i whenever there exists an edge of F_i connecting a vertex of C with a vertex of C'. Let $f_i : V(G) \to V(G/F_i)$ be the natural projection, that is, $u \in V(G)$ is mapped to the component of $G \setminus F_i$ to which it belongs. Then

$$f = (f_1, \ldots, f_p) : G \to G/F_1 \,\square\, \cdots \,\square\, G/F_p \tag{1}$$

is an induced embedding of G. Moreover, by adding edges to each factor G/F_i to make it complete, the embedding f is still induced. It follows that f can be considered as an induced embedding of G into a Hamming graph. In addition, f is an *irredundant embedding* meaning that each G/F_i has at least two vertices and each vertex of it appears as a coordinate in some image of a vertex of G. (To obtain an induced embedding of G into a Cartesian product (of factors that are not necessarily complete), Condition B must be modified, see [22].)

We will make use of the following additional properties of a labeling that fulfills Condition B, see [18, Lemmas 3.1 and 3.2]:

(i) in an induced cycle of length > 3, every label must appear at least twice, and

(ii) if every induced path between two vertices contains labels i and j, then every path between these two vertices contains these two labels.

In addition, it is easy to see that if a maximal part of an induced cycle C is labeled alternatively with i and j, then i and j must also exist on the other part of C. In particular, if we have the sequence iji on C, then i appears at least once more on C.

We now turn our attention to Sierpiński graphs. Every S_k^n can be embedded in a Hamming graph with two factors as follows. Label the clique and the non-clique edges of S_k^n with labels p and q, respectively. Call this labeling a *$p|r$-labeling*. Clearly, a $p|r$-labeling fulfills Condition A. Moreover, since no two non-clique edges are incident, Condition B holds as well.

Let $k \geq 3$. Then the *Sierpiński triangle labeling* of S_k^n is inductively defined as follows. Label the edges of $S_k^1 \cong K_k$ with label 1. Suppose now S_k^n, $n \geq 1$, has already been labeled. Then label every subgraph iS_k^n $(1 \leq i \leq k)$ of S_k^{n+1} identically as S_k^n and label the edges $e_{ij}^{(n+1)}$ with label $n+1$. Clearly, the Sierpiński triangle labeling of S_k^n uses n labels. Note also that the Sierpiński triangle labeling of S_k^2 coincides with its $1|2$-labeling.

Theorem 3.2 *Let $k \geq 3$ and $n \geq 1$. Then the Sierpiński triangle labeling of S_k^n yields an induced embedding*

$$S_k^n \to \widehat{S_k^n} \,\square\, \widehat{S_k^{n-1}} \,\square\, \cdots \,\square\, \widehat{S_k^1}\,.$$

Proof. Let $k \geq 3$ be a fixed integer. The Sierpiński triangle labeling clearly fulfills Condition A. Let u, v be two non-adjacent vertices of S_k^n. Consider a shortest path P between u and v and let i be the largest label on P. Then $i > 1$ and every induced path between u and v contains labels 1 and i. Hence Condition B is fulfilled and thus the embedding (1) can be used.

Let F_i, $1 \leq i \leq n$, be the set of edges of S_k^n labeled with $n-i+1$ in the Sierpiński triangle labeling of S_k^n. We are going to prove that for any $n \geq 1$ and for any $1 \leq i \leq n$, $S_k^n/F_i = \widehat{S_k^i}$.

Let $n = 1$. Then $S_k^1 = K_k$ and all of its edges are labeled with 1. Hence $\widehat{S_k^1} = K_k = S_k^1/F_1$. Suppose Theorem 3.2 holds for some $n \geq 1$ and consider S_k^{n+1}. Since $F_1 = \{e_{ij}^{(n+1)} \mid i \neq j\}$ we infer that $S_k^{n+1}/F_1 = K_k = \widehat{S_k^1}$. Let next $i \geq 2$. Then every edge of F_i lies in some subgraph jS_k^n. Let jF_i be the restriction of F_i to jS_k^n and note that jF_i coincides with the labeling as F_{i-1} in S_k^n. Hence, by the induction hypothesis, it follows that $jS_k^n/jF_i = \widehat{S_k^{i-1}}$. But then $S_k^{n+1}/F_i = \widehat{S_k^i}$ by the way the generalized Sierpiński triangle graphs are constructed. $\square$

4 A lower bound on $\mathrm{Hdim}(S_3^n)$

In this section we prove:

Theorem 4.1 *For any $n \geq 2$,*

$$\mathrm{Hdim}(S_3^n) \geq \frac{7}{4} \cdot 3^{n-3} + 3 \cdot 2^{n-4} + \frac{3}{2}n - \frac{9}{4}\,.$$

To prove the theorem we construct a *merging labeling* of S_3^n, $n \geq 2$, as follows. For $n = 2$, label every edge of iS_3^1 with i and for any $j \neq k$, label the edge $e_{jk}^{(2)}$ with i, where $\{i, j, k\} = \{1, 2, 3\}$. Proceed by induction on n as follows. Label every iS_3^{n-1} with the same pattern as S_3^{n-1}, but such that iS_3^{n-1} and jS_3^{n-1} use pairwise different labels for any $i \neq j$. In addition, label the edges $e_{12}^{(n)}$, $e_{23}^{(n)}$, and $e_{13}^{(n)}$ with the same labels as $3e_{12}^{(n-1)}$, $1e_{23}^{(n-1)}$, and $2e_{13}^{(n-1)}$, respectively. Note that this labeling does not fulfill Condition B since some labels appears only once at $C_{123}^{(n)}$.

We thus need to merge every label that appears only once on $1P_{23}^{(n-1)}$, only once on $2P_{13}^{(n-1)}$, and only once on $3P_{12}^{(n-1)}$ with the exception of the edges $1e_{23}^{(n-1)}$, $2e_{13}^{(n-1)}$, and $3e_{12}^{(n-1)}$, respectively. The merging is done as follows. Consider the following pairs of oriented subpaths of $C_{123}^{(n)}$: $12P_{23}^{(n-2)}, 32P_{21}^{(n-2)}$; $13P_{23}^{(n-2)}, 23P_{13}^{(n-2)}$; and $31P_{12}^{(n-2)}, 21P_{13}^{(n-2)}$. Here oriented means that each of these paths has it start and its end, for instance, $12P_{23}^{(n-2)}$ starts in $\langle 122\ldots 2\rangle$ and ends in $\langle 1233\ldots 3\rangle$. Now traverse $12P_{23}^{(n-2)}$ and $32P_{21}^{(n-2)}$ in parallel. As soon as a label ℓ_1 is found on $12P_{23}^{(n-2)}$ that appears only once on $1P_{23}^{(n-1)}$, merge it with the corresponding label ℓ_3 of $32P_{21}^{(n-2)}$. (Note that ℓ_3 also appears only once on $3P_{21}^{(n-1)}$ by the construction.) More precisely, we replace every label ℓ_3 in S_3^n with ℓ_1. Do the same procedure for the other two pairs of paths. An example of a merging labeling of S_3^5 is shown in Fig. 3.

Proposition 4.2 *A merging labeling of S_3^n, $n \geq 2$, fulfills Conditions A and B.*

Proof. Edges that form a triangle are labeled with the same label, hence Condition A is fulfilled. Note also that Condition B is fulfilled on S_3^2. Let now $n > 2$ and let u, v be vertices of S_3^n with $d(u, v) \geq 2$. Let p be the smallest in the sense that both u and v are in $i_1 i_2 \ldots i_p S_3^{n-p}$. Then $p < n - 1$ since $d(u, v) \geq 2$. Let $u \in i_1 i_2 \ldots i_p j_1 S_3^{n-p-1}$, $v \in i_1 i_2 \ldots i_p j_2 S_3^{n-p-1}$, and let $\{j_1, j_2, j_3\} = \{1, 2, 3\}$.

Let P be a shortest u, v-path. Suppose first that P contains the edges $i_1 i_2 \ldots i_p e_{j_1 j_3}^{(n-p)}$ and $i_1 i_2 \ldots i_p e_{j_2 j_3}^{(n-p)}$. Then the labels of these two edges are on any induced u, v-path by the way the merging labeling is constructed. In the other case, P contains a unique edge of the form $e = i_1 i_2 \ldots i_p e_{rq}^{(n-p)}$, namely the edge $i_1 i_2 \ldots i_p e_{j_1 j_2}^{(n-p)}$. By the same argument its label appears on every induced u, v-path. Since $d(u, v) \geq 2$, the edge e has at least one incident edge on P, say f. We may assume without loss of generality that $f \in$

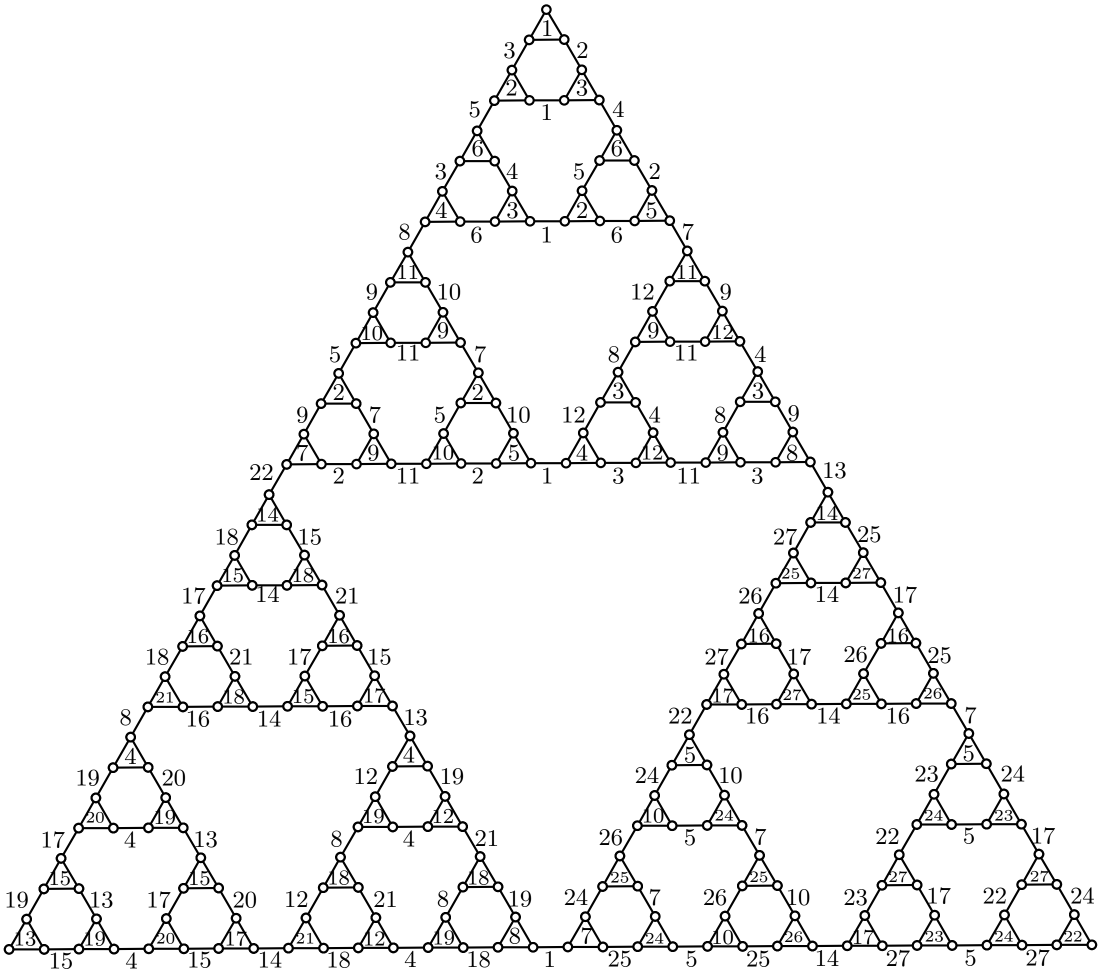

Figure 3: A merging labeling of S_3^5

$i_1 i_2 \dots i_p j_2 S_3^{n-p-1}$. Then the label of f appears also on the triangle of $i_1 i_2 \dots i_p j_3 S_3^{n-p-1}$ that is incident with the edge $i_1 i_2 \dots i_p e_{j_1 j_3}^{(n-p)}$. Again by the construction, the label of f appears on any induced u,v-path. $\square$

Lemma 4.3 *Let S_3^n, $n \geq 2$, be labeled with a merging labeling. Then every label of a non-clique edge of $P_{ij}^{(n)}$, $i,j \in \{1,2,3\}$, besides $e_{ij}^{(n)}$, appears exactly twice on $P_{ij}^{(n)}$.*

Proof. There is nothing to be proved for $n = 2$. We can restrict to $P_{23}^{(n)}$ by symmetry. Note that the labels of the edges $2e_{23}^{(n-1)}$ and $3e_{23}^{(n-1)}$ are merged in S_3^n and have thus the same label. Hence every label of a non-clique edge of $P_{ij}^{(n)}$, $i, j \in \{1,2,3\}$, besides $e_{ij}^{(n)}$, appears at least twice on $P_{ij}^{(n)}$ by induction.

It remains to prove that no non-clique edge appears more than twice. This clearly holds for $n = 3, 4$, cf. Fig. 3. Let now $n \geq 5$. Note first that the assertion holds for the label of $2e_{23}^{(n-1)}$ and $3e_{23}^{(n-1)}$. Indeed, their labels were unique on $2P_{23}^{(n-1)}$ and $3P_{23}^{(n-1)}$, respectively and were henceforth merged in the last step of the construction. The label of the edges $22e_{23}^{(n-2)}$ and $23e_{23}^{(n-2)}$ (which is the same) appears only once on $2P_{13}^{(n-1)}$ and is also merged in S_3^n. But this label appears on $23P_{13}^{(n-2)}$ and is merged with a label from $13P_{23}^{(n-1)}$. In other words, this label does not appear in $3S_3^{n-1}$ and consequently not on $3P_{23}^{(n-1)}$. By symmetry, the assertion also holds for the label of the edges $32e_{23}^{(n-2)}$ and $33e_{23}^{(n-2)}$.

Next we show that the label ℓ of non-clique edges $222e_{23}^{(n-3)}$ and $223e_{23}^{(n-3)}$ appears twice on $2P_{13}^{(n-1)}$ and is not merged in S_3^n. Clearly ℓ appears once on $223P_{13}^{(n-3)}$ (on the edge incident with $\langle 22311\ldots 1\rangle$) and was in $2S_3^{n-1}$ merged with the label of the edge on $213P_{23}^{(n-3)}$ incident with $\langle 21322\ldots 2\rangle$. This label is in $21S_3^{n-2}$ present also on the edges $211e_{13}^{(n-3)}$ and $213e_{13}^{(n-3)}$, which are both on $2P_{13}^{(n-1)}$.

Similarly, the label ℓ' of the edges $232e_{23}^{(n-3)}$ and $233e_{23}^{(n-3)}$ appears twice on $2P_{13}^{(n-1)}$ and is not merged in S_3^n. Clearly ℓ' appear once on $2P_{13}^{(n-1)}$ since it is in the triangle of the extreme vertex $\langle 23311\ldots 1\rangle$ in $233S_3^{n-3}$. But ℓ' is also in the triangle of the extreme vertex $\langle 23211\ldots 1\rangle$ in $232S_3^{n-3}$. Hence it was merged in $2S_3^{n-1}$ with the label of the triangle of the extreme vertex $\langle 21333\ldots 3\rangle$ in $212S_3^{n-3}$. But this was again merged in $21S_3^{n-2}$ with the label of the triangle of the extreme vertex $\langle 21133\ldots 3\rangle$, which lie on $2P_{13}^{(n-1)}$.

The conclusion also holds for the labels of $P_{23}^{(n)}$ in $3S_3^{n-1}$ that are symmetric to the edges in previous two paragraphs.

Finally, for all the other non-clique edges of $P_{23}^{(n)}$ the statement follows by induction. □

Next we calculate the number of labels of a merging labeling of S_3^n. Let b_n be the number of labels different from 1 that appear on $P_{23}^{(n)}$ exactly once. In other words, b_n is the number of labels of $1S_3^n$ that will be merged with some other label in S_3^{n+1}. (Clearly label 1 will not be merged.) Hence

$$b_n = 2b_{n-1} - 2c_n\,,$$

where c_n represents the number of labels that appear twice on $P_{23}^{(n)}$ for the first time. To determine c_n, Lemma 4.3 implies that we only need to find clique edges whose labels appear twice on $P_{23}^{(n)}$ for the first time and, moreover, one edge must be in $2S_3^{n-1}$ and the second one in $3S_3^{n-1}$. By the way merging is defined this can happen if the first edge is in $223S_3^{n-3}$ and its label appears on both $22P_{23}^{(n-2)}$ and $22P_{13}^{(n-2)}$ exactly once. The label of such an edge is then merged with the label of some edge in $213S_3^{n-3}$ that again appears on $21P_{23}^{(n-2)}$ and $21P_{13}^{(n-2)}$ exactly once. The edge on $21P_{13}^{(n-2)}$ is then on $C_{123}^{(n)}$ and its label is merged with the label of an edge in $312S_3^{n-3}$ that appears on $31P_{12}^{(n-2)}$ and $31P_{23}^{(n-2)}$ exactly once by symmetry. Finally this was merged with a label in $332S_3^{n-3}$ that again appears only once on $33P_{12}^{(n-2)}$ and $33P_{23}^{(n-2)}$. Loking to Fig. 3 we infer that that $c_4 = 1$ (label 9) and $c_5 = 1$ (label 17).

Hence we need to treat clique edges on $223P_{23}^{(n-3)}$. For this sake we define even and odd clique edges of $P_{23}^{(n)}$ as follows. Let $T_1, T_2, \ldots, T_{2^{n-1}}$ be the consecutive triangles with edges in $P_{23}^{(n)}$. (On Fig. 3, triangle T_1 is labeled with 13, and T_{16} with 22.) Then we say that a clique edge $e \in P_{23}^{(n)}$ is *even/odd* if $e \in T_i$ and i is even/odd. Note that the label of an odd clique edge from $223P_{23}^{(n-3)}$ appears twice on $22P_{13}^{(n-2)}$. Hence it appears twice on $2C_{123}^{(n-1)}$ and is not merged at this step. So we only need to consider even clique edges from $223P_{23}^{(n-3)}$. We will show by induction that $c_n = n-4$ for $n \geq 5$. Note that for $n = 5$ there is only one such label, namely label 17 on Fig. 3. For S_3^n, $n > 5$, every even clique edge of $2233P_{23}^{(n-4)}$ has this property as well as the even clique edge of $T_{3\cdot 2^{n-5}}$. Hence $c_n = n - 4$ for $n \geq 5$.

Returning back to b_n we now have:

$$b_n = 2b_{n-1} - 2(n-4),\ b_5 = 10\,,$$

which solves to

$$b_n = 2^{n-3} + 2n - 4,\ n \geq 5\,.$$

Note that this formula holds also for $n = 4$.

Let finally a_n, $n \geq 4$, be the number of labels in a merging labeling of S_3^n. Then

$$a_n = 3a_{n-1} - \frac{3}{2}b_{n-1} = 3a_{n-1} - \frac{3}{2}(2^{n-4} + 2n - 6), a_4 = 12$$

since we merge six parts into three by pairs. The theorem now easily follows. (We need to check $n = 2, 3$ separately.)

5 An upper bound on $\mathrm{Hdim}(S_k^n)$

In this section we prove an upper bound on the Hamming dimension of S_k^n for $k \geq 3$. We first establish some exact values.

Proposition 5.1 *(i)* $\mathrm{Hdim}(S_3^2) = 3$, $\mathrm{Hdim}(S_3^3) = 6$.
(ii) For any $k \geq 4$, $\mathrm{Hdim}(S_k^2) = 2$.

Proof. (i) By Theorem 4.1, $\mathrm{Hdim}(S_3^2) \geq 3$. That $\mathrm{Hdim}(S_3^2) \leq 3$ follows by the fact that on the cycle $C_{123}^{(2)}$ of S_3^2 each label appears at least twice. Note that the merging labeling is the unique 3-labeling of S_3^2 that satisfies Conditions A and B.

Using Theorem 4.1 again we have $\mathrm{Hdim}(S_3^3) \geq 6$. Since $C_{123}^{(3)}$ has length 12 (and every label of an induced cycle must appear at least twice on it), there can be at most 6 different labels on $C_{123}^{(3)}$. If for $\{i,j,\ell\} = \{1,2,3\}$ every $\ell P_{ij}^{(2)}$ contains three labels in ℓS_3^2, then each ℓS_3^2 contains the same three labels as $\ell P_{ij}^{(2)}$ (because the merging labeling is the unique appropriate 3-labeling of S_3^2). Such a labeling thus uses at most 6 different labels. Similarly, if some $\ell P_{ij}^{(2)}$ contains only two different labels we infer that only these two labels can be used on ℓS_3^2.

(ii) Let $k \geq 4$. We claim that the $1|2$-labeling of S_k^2 yields a unique induced embedding of S_k^2 into a Hamming graph and hence $\mathrm{Hdim}(S_k^2) = 2$.

Since S_k^2 is not a complete graph we need at least two labels. By Condition A, all edges of iS_k^1, $i = 1, 2, \ldots, k$, must receive the same label. By Condition B, every edge $e_{ij}^{(2)}$, $j \neq i$, must have different label from the labels of iS_k^1 and jS_k^1. If all iS_k^1 have the same label, then the non-clique edges of any cycle $C_{pqr}^{(2)}$ must have the same label, for otherwise one label appears only once on $C_{pqr}^{(2)}$. Since p, q, and r are arbitrary we obtain the $1|2$-labeling.

Suppose next that two of iS_k^1, $i = 1, 2, \ldots, k$, are labeled with 1 and among the others at least one with 2. We may choose the notation so that $1S_k^1$ and $2S_k^1$ have label 1 and $3S_k^1$ label 2. Then by Condition B, edges $e_{12}^{(2)}$, $e_{13}^{(2)}$, and $e_{23}^{(2)}$ cannot have label 1, moreover $e_{13}^{(2)}$ and $e_{23}^{(2)}$ cannot have label 2 by the same condition. But then $e_{12}^{(2)}$ must have label 2, for otherwise we have the same contradiction as above in $C_{123}^{(2)}$. Consider now vertices $\langle 13 \rangle$ and $\langle 23 \rangle$ to find the final contradiction with Condition B.

Assume finally that all the iS_k^1, $i = 1, 2, \ldots, k$, have different labels, say iS_k^1 has label i. To satisfy Condition B, the edge $e_{12}^{(2)}$ of $C_{123}^{(2)}$ must have label 3, $e_{13}^{(2)}$ label 2, and $e_{23}^{(2)}$ label 1. By the same argument applied on $C_{124}^{(2)}$, the edge $e_{12}^{(2)}$ must have label 4, a final contradiction. $\square$

We are now ready for the main result of this section.

Theorem 5.2

$$
\begin{aligned}
(i) \quad & \mathrm{Hdim}(S_3^n) & \leq \quad & 5 \cdot 3^{n-3} + 1 \quad (n \geq 3)\,. \\
(ii) \quad & \mathrm{Hdim}(S_k^n) & \leq \quad & \frac{2}{k-1} k^{n-2} + \frac{2k-4}{k-1} \quad (n \geq 2)\,.
\end{aligned}
$$

Proof. Labels that appear in more than one iS_k^{n-1} will be called *common labels*.

For a fixed k and $n \geq 3$, consider a labeling of S_k^n that fulfills Conditions A and B and uses $\mathrm{Hdim}(S_k^n)$ labels. This labeling has at most $\mathrm{Hdim}(S_k^{n-1})$ different labels in each subgraph iS_k^{n-1} (because iS_k^{n-1} is isomorphic to S_k^{n-1}). In addition, by Condition B, there must be at least two labels in each iS_k^{n-1} that appear also in $S_k^n \backslash iS_k^{n-1}$. Hence we get

$$\mathrm{Hdim}(S_k^n) \geq k(\mathrm{Hdim}(S_k^{n-1}) - 2) + \alpha_n \,,$$

where α_n denotes the maximum number of common labels. Setting

$$a_n = k(a_{n-1} - 2) + \alpha_n \,,$$

we thus have $\mathrm{Hdim}(S_k^n) \leq a_n$.

Consider iS_k^{n-1} and $C_{ij\ell}^{(n)}$. For the closest vertices of $e_{ij}^{(n)}$ and $e_{i\ell}^{(n)}$ on $C_{ij\ell}$ we observe that by Condition B we need (at least) two labels of iS_k^{n-1} on the other part of $C_{ij\ell}^{(n)}$. Hence for every $i = 1, 2, \ldots, k$ there are at most $a_{n-1} - 2$ labels that appear only in iS_k^{n-1}. First we assume that the maximum number of labels is attained when we have $a_{n-1} - 2$ different labels in every iS_k^{n-1}. Even more, these two labels cannot be on $e_{ij}^{(n)}$ or $e_{i\ell}^{(n)}$, since otherwise we can include these two edges and consider the other two vertices of $e_{ij}^{(n)}$ and $e_{i\ell}^{(n)}$. Thus we have 6 positions on $C_{ij\ell}^{(n)}$ for new labels in iS_k^{n-1}, jS_k^{n-1}, and ℓS_k^{n-1}, and additional 3 edges $e_{ij}^{(n)}$, $e_{i\ell}^{(n)}$ and $e_{j\ell}^{(n)}$—all together 9 positions. By the above argument, each position in iS_k^{n-1}, jS_k^{n-1}, and ℓS_k^{n-1} may contain more than one edge but all such edges can be viewed just as one. But then in $C_{ij\ell}^{(n)}$ we may have at most $4 = \left\lfloor \frac{9}{2} \right\rfloor$ common labels.

Suppose now that we can use 5 common labels. First we consider a longer path $P_{ij\ell}$ between $\langle i\ell\ell \ldots \ell \rangle$ and $\langle j\ell\ell \ldots \ell \rangle$ in $C_{ij\ell}$ for every i, j, and ℓ. If every $C_{ij\ell}$ contains at most two common labels, $P_{ij\ell}$ clearly contains both labels. But then $P_{ijr} = P_{ij\ell}$ for every $r \notin \{i, j, \ell\}$ and every C_{ijr} contains these two labels. This is a contradiction since we have used 5 common labels. Next suppose that every $C_{ij\ell}$ contains at most 3 common labels. If $P_{i\ell j}$ contains only two of these labels, then both $P_{ij\ell}$ and $P_{j\ell i}$ contain all three. Again $P_{ijr} = P_{ij\ell}$ for every $r \notin \{i, j, \ell\}$ and every C_{ijr} contains these three labels—a contradiction. Next suppose that $C_{ij\ell}$ contains four common labels. If $P_{ij\ell}$ contains only three common labels, we have only 4 positions in $C_{ij\ell} - P_{ij\ell}$ and one label, say 4 is present only on $C_{ij\ell} - P_{ij\ell}$. By the above, both $e_{i\ell}^{(n)}$ and $e_{j\ell}^{(n)}$ must have label 4. The label of $e_{ij}^{(n)}$, say 3, must be in ℓS_k^{n-1} together with a common label 2. Label 2 must also be in one of iS_k^{n-1} or jS_k^{n-1}. We may assume that it is in iS_k^{n-1} (together with label 1). Hence $P_{i\ell j}$ contains four common labels. If label 5 exists in rS_k^{n-1}, $r \notin \{i, j, \ell\}$, then $C_{i\ell r}$ contains 5 common labels which is not possible. Hence let $e_{pr}^{(n)}$ have label 5. If $p \in \{i, \ell\}$ (or by symmetry $r \in \{i, \ell\}$) then $C_{i\ell r}$ (or $C_{i\ell p}$) contains 5 common labels again. If finally $p, r \notin \{i, j, \ell\}$, either $e_{pi}^{(n)}$ or $e_{ri}^{(n)}$ have label 5 which is not possible. Thus $\alpha_n \leq 4$, hence

$$a_n = k(a_{n-1} - 2) + 4 \,, a_3 = 4 \,.$$

By Proposition 5.1, the initial conditions for $k = 3$ and $k \geq 4$ are $\mathrm{Hdim}(S_3^3) = 6$ and $\mathrm{Hdim}(S_k^2) = 2$, respectively. Solving the recurrence yields the result. $\square$

Corollary 5.3 *For any $k \geq 4$, $\mathrm{Hdim}(S_k^3) = 4$.*

Proof. By Theorem 5.2, $\mathrm{Hdim}(S_k^3) \leq 4$. A 4-labeling of S_k^3 that satisfies Conditions A and B can be constructed as follows. Use the 1|2-, 2|3-, 3|4-, and 4|1-labelings on $1S_k^2$, $2S_k^2$, $3S_k^2$, and $4S_k^2$, respectively. Label the edges $e_{12}^{(3)}$, $e_{23}^{(3)}$, $e_{34}^{(3)}$, and $e_{14}^{(3)}$ with 4, 1, 2, and 3, respectively. Next, we may choose labels 2 or 4 for the edge $e_{13}^{(3)}$ and labels 1 or 3 for the edge $e_{24}^{(3)}$. Finally, for every $i \in \{5, 6, \dots, k\}$ use the 1|3-labeling on iS_k^2, label edges $e_{i1}^{(3)}$ and $e_{i2}^{(3)}$ with 4, edges $e_{i3}^{(3)}$ and $e_{i4}^{(3)}$ with 2, and all the other edges $e_{ij}^{(3)}$, $j \in \{5, 6, \dots, k\}$, $i \neq j$, with 2. For this labeling, Condition A clearly holds. Moreover, a straightforward checking on cycles $C_{pqr}^{(3)}$ shows that Condition B is fulfilled for it as well. $\square$

Note that in Theorem 5.2 the equality holds for S_k^2 and S_k^3, $k \geq 4$. The upper bound (ii) is also exact for S_4^4. Indeed, the bound is 12, and on the other hand, two different appropriate labelings of S_4^4 are shown in Fig. 5.

6 Isometric embedding

In this final section we consider isometric embeddings of S_k^n into Cartesian product graphs. In this case the classical theory due to Graham and Winkler asserts that there is a unique such embedding that is irredundant and has the largest number of factors. The embedding is described in many papers and books, see for instance [8], and is called the *canonical isometric representation*. We recall that it is defined just as the embedding f was introduced in Section 3 where the partition of the edge set of G is done with respect to the transitive closure Θ^* of the relation Θ. Here edges $e = xy$ and $f = uv$ of G are in relation Θ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$. The canonical isometric representation is *trivial* if G contains only one Θ^* class.

It is easy to see that no two edges of a geodesic are in relation Θ, a fact that will be used later. We will also need the following well-known lemma, cf. [8]:

Lemma 6.1 *Suppose P is a walk connecting the endpoints of an edge e. Then P contains an edge $f \neq e$ with $e\Theta f$.*

Now we have:

Proposition 6.2 *Let $k \geq 4$. Then for any $n \geq 1$ the canonical isometric representation of S_k^n is trivial.*

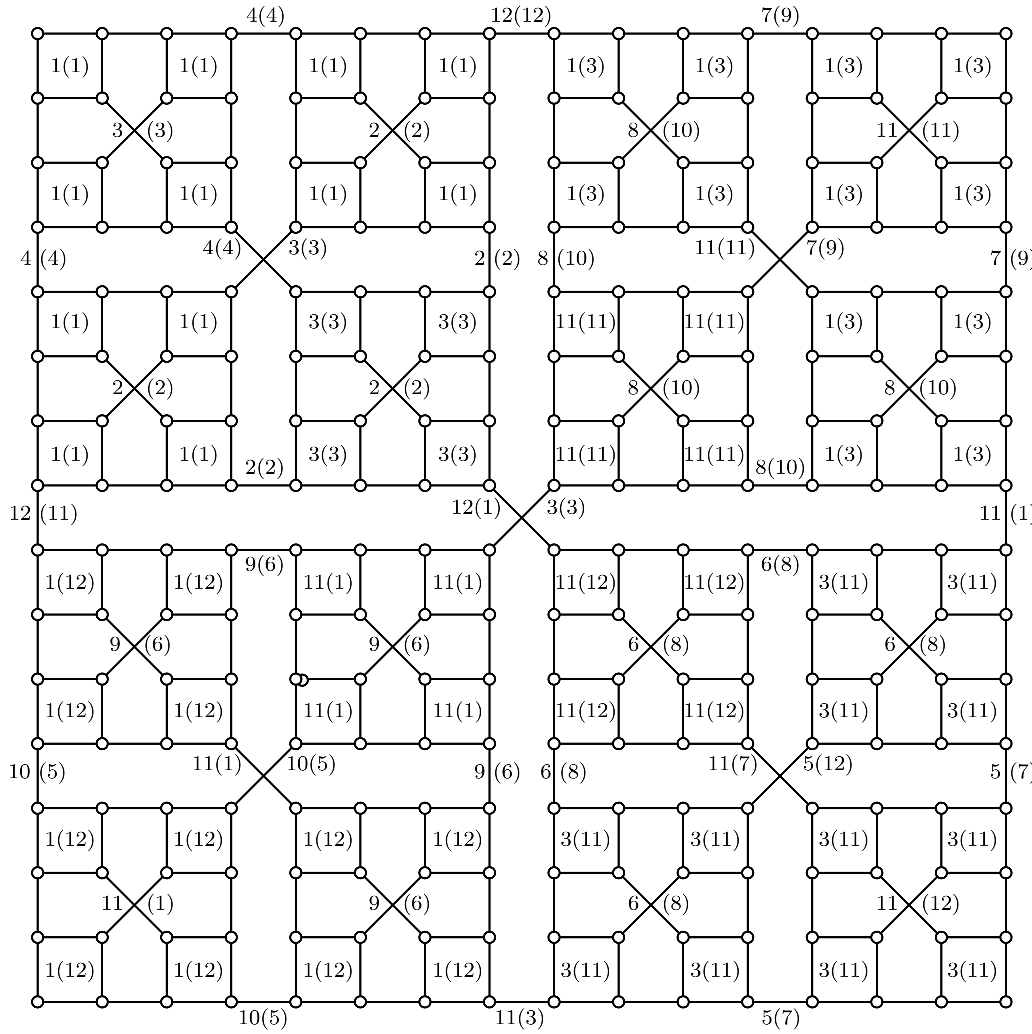

Figure 4: Two labelings of S_4^4

Proof. For a given $k \geq 4$ we proceed by induction on n. Graph S_k^1 is isomorphic to K_k, hence the assertion clearly holds in this case. Let $n > 1$. Then for $i = 1, \ldots, k$, the subgraph iS_k^{n-1} contains a single Θ^*-class by the induction assumption. For $i = 3, 4, \ldots, k$, let $C_{12i}^{(n)}$ be a shortest cycle containing the edges $e_{12}^{(n)}$, $e_{1i}^{(n)}$, and $e_{2i}^{(n)}$. Then Lemma 6.1 implies that $C_{12i}^{(n)}$ contains an edge f with $f \Theta e_{12}^{(n)}$. Moreover, f can only lie in iS_k^{n-1}. Hence the edges of iS_k^{n-1}, $i \geq 3$, all lie in the same Θ^*-class. By the symmetry of S_k^n, the canonical isometric representation of S_k^n is then trivial. $\square$

By Proposition 6.2 we may hope for a nontrivial isometric representation of S_k^n only when $k = 3$. This is indeed the case as the main result (Theorem 6.5) of this section asserts. We need some preparation for it.

Proposition 6.3 *Let $n \geq 1$ and let F be a Θ^*-class of S_3^n. Then $|P_{ij}^{(n)} \cap F| \geq 1$ for $i \neq j$.*

Proof. The statement is clearly true for $n = 1$. Let $n > 1$ and let F be an arbitrary Θ^*-class of S_3^n. If $|F \cap iS_3^{n-1}| \geq 1$, then by the induction hypothesis (applied to iS_3^{n-1}), F intersects shortest paths $iP_{ij}^{(n-1)}$, $iP_{i\ell}^{(n-1)}$, and $iP_{j,\ell}^{(n-1)}$ for $\{i, j, \ell\} = \{1, 2, 3\}$. Let e be in $iP_{j,\ell}^{(n-1)} \cap F$. If the antipodal edge of e on $C_{123}^{(n)}$ is $e_{j\ell}^{(n)}$, we are done since $e_{j\ell}^{(n)}$ is on $P_{j,\ell}^{(n)}$. Otherwise, the antipodal edge of e on $C_{123}^{(n)}$ is either on $jP_{i\ell}^{(n-1)}$ or $\ell P_{ij}^{(n-1)}$. Induction completes the proof. $\square$

It is well-known (and easy to prove) that edges from different blocks of a graph are not in relation Θ and hence also not in relation Θ^*. For our purposes we need the following modification of this fact.

Lemma 6.4 *Let H be isometric subgraph of G and let e and f be edges from different blocks of H. Then e is not in relation Θ with f in G.*

Proof. Let $e = uv$ and $f = xy$. By the above fact, e and f are not in relation Θ in H, that is,

$$d_H(u, x) + d_H(v, y) = d_H(u, y) + d_H(v, x)\,.$$

Since H is an isometric subgraph of G, it follows that

$$d_G(u, x) + d_G(v, y) = d_G(u, y) + d_G(v, x)\,,$$

hence e and f are not in relation Θ in G. $\square$

Note that we cannot conclude in Lemma 6.4 that e and f are not in relation Θ^* in G. For instance, consider P_3 as a subgraph of $K_{2,3}$. Then it is isometric in $K_{2,3}$ yet its edges are in relation Θ^*.

To describe Θ^*-classes of S_3^n, let $\{i, j, k\} = \{1, 2, 3\}$ and set

$$F_n^i = \{\langle i^n \rangle \langle i^{n-1} j \rangle, \langle i^n \rangle \langle i^{n-1} k \rangle\} \cup \{e_{jk}^{(\ell)} \mid \ell = 1, 2, \ldots, n\}\,.$$

Note that $|F_n^i| = n + 2$.

Now we cab state the main result of this section:

Theorem 6.5 *Let $n \geq 2$. Then the Θ^*-classes of S_3^n are F_n^1, F_n^2, F_n^3, and $\widetilde{F_n} = E(S_3^n) \setminus (F_n^1 \cup F_n^2 \cup F_n^3)$.*

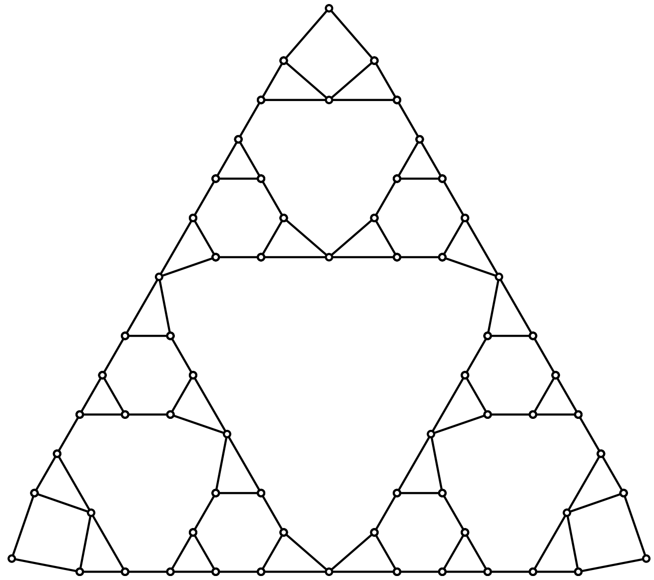

Figure 5: The factor graph $S_3^4/\widetilde{F_4}$

Proof. It is straightforward to check the result for $n=2$, where $\widetilde{F_3}=\emptyset$ so that in this case we have three Θ^*-classes.

Let $i \in \{1,2,3\}$ and consider F_n^i. By induction assumption (and the fact that iS_3^{n-1} is an isometric subgraph of S_3^n), we infer that $\langle i^n\rangle\langle i^{n-1}j\rangle, \langle i^n\rangle\langle i^{n-1}k\rangle \in F_n^i$, as well as $e_{jk}^{(\ell)} \in F_n^i$ for $\ell = 1,2,\ldots,n-1$. Moreover, the edge $e_{jk}^{(n)}$ belongs to F_n^i because it is the antipodal edge of $e_{jk}^{(n-1)}$ on $C_{123}^{(n)}$. (Recall that $C_{123}^{(n)}$ is the shortest cycle containing the edges $e_{12}^{(n)}$, $e_{23}^{(n)}$, and $e_{31}^{(n)}$.) Hence the edges of F_n^i belong to a common Θ^*-class. It remains to show that (i) no two edges from different sets F_n^1, F_n^2, F_n^3, and $\widetilde{F_n}$ are in relation Θ and that (ii) in $\widetilde{F_n}$ any two edges are in relation Θ^*.

For assertion (i), by symmetry it suffices to prove that no edge of F_n^1 is in relation Θ with any other edge. Moreover, denoting with G_2 and G_3 the connected components of $S_3^n \setminus F_n^1$, where $\langle 2^n\rangle \in G_2$, it suffices (using symmetry again) to prove that no edge of F_n^1 is in relation Θ with an edge of G_2.

Note first that G_2 is isometric in S_3^n. Moreover, the graph induced with $V(G_2)$ and vertices $\langle 1^n\rangle$ and $\langle 1^{n-1}3\rangle$ is also isometric in S_3^n. Then Lemma 6.4 implies that none of the edges $\langle 1^n\rangle\langle 1^{n-1}2\rangle$, $\langle 1^n\rangle\langle 1^{n-1}3\rangle$, and $\langle 1^{n-1}2\rangle\langle 1^{n-1}3\rangle$ is in relation Θ with no edge in G_2. Let $\ell \in \{0,\ldots,n-2\}$ and consider the subgraph of S_3^n induced by G_2 and $\langle 132^{n-\ell-1}\rangle$. We infer again that this subgraph is isometric, hence applying Lemma 6.4 we conclude that $\langle 1^{n-1}2\rangle\langle 1^{n-1}3\rangle$ is in relation Θ with no edge of G_2. This proves (i).

It remains to prove that any two edges of $\widetilde{F_n}$ are in relation Θ^*. If $n=3$, it is straightforward to check that $\langle 112\rangle\langle 121\rangle\Theta\langle 322\rangle\langle 321\rangle\Theta\langle 122\rangle\langle 123\rangle$. By symmetry and transitivity the result follows. Let $n \geq 4$. Then because $C_{123}^{(n)}$ is isometric,

$$\langle 12^{n-1}\rangle\langle 12^{n-2}3\rangle\Theta\langle 321^{n-2}\rangle\langle 321^{n-3}2\rangle$$

as well as

$$\langle 12^{n-2}3\rangle\langle 12^{n-3}32\rangle\Theta\langle 321^{n-3}2\rangle\langle 321^{n-4}21\rangle\,.$$

Now apply induction, symmetry, and transitive closure to conclude that $\widetilde{F_n}$ is indeed a Θ^*-class. $\square$

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