Arc-transitive abelian regular covers of the Heawood graph

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Abstract

In this sequel to the paper 'Arc-transitive abelian regular covers of cubic graphs', all arc-transitive abelian regular covers of the Heawood graph are found. These covers include graphs that are 1-arc-regular, and others that are 4-arc-regular (like the Heawood graph). Remarkably, also some of these covers are 2-arc-regular.

1 Introduction

This is a sequel to the paper 'Arc-transitive abelian regular covers of cubic graphs' by the same authors [4], in which a new approach was introduced for finding arc-transitive abelian regular covers of a given finite symmetric cubic graph, and applied to find all such covers of K_4, $K_{3,3}$, the cube Q_3, and the Petersen graph. In this paper, we do the same for the Heawood graph.

A significant amount of background material was given in [4], but we summarise some of the important details here, before proceeding.

First, a graph Y is called a *cover* of a another graph X if there exists a surjective mapping $p\colon V(Y) \to V(X)$ which preserves adjacency and is also locally bijective (preserving valence at each vertex). Any such p is called a *covering projection.* Second, an *automorphism* of a graph X is a bijective graph homomorphism from X to X, and under composition, the automorphisms of X form a group, called the *automorphism* *group* of X and denoted by $\mathrm{Aut}\,X$. The connected graph X is called *symmetric* (or *arc-transitive*) if $\mathrm{Aut}\,X$ is transitive on the arcs (ordered edges) of X. Also a subgroup L of $\mathrm{Aut}\,X$ is called *semi-regular* if the stabilizer in L of every vertex or arc of X is trivial — that is, if L acts regularly on each of its orbits on vertices and arcs of X.

If $p\colon Y \to X$ is a covering projection, then an automorphism α of X is said to *lift* to an automorphism β of Y if $\alpha \circ p = p \circ \beta$. The set of all lifts of the identity automorphism of X is called the *group of covering transformations*, or *voltage group*, and is sometimes denoted by $\mathrm{CT}(p)$.

For any semi-regular subgroup N of $\mathrm{Aut}\,X$, we may define a quotient graph X_N by taking vertices as the orbits of N on $V(X)$ and edges as the orbits of N on $E(X)$, with the obvious incidence. A covering projection $p : Y \to X$ is called *regular* if there exists a semi-regular subgroup N of $\mathrm{Aut}\,Y$ such that the quotient graph Y_N is isomorphic to X. In that case, we call Y a *regular cover* of X, with covering transformation group N; also Y is called an abelian cover (and p an abelian covering projection) if the group N is abelian.

Next, an *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such any two consecutive v_i are adjacent, and any three consecutive v_i are distinct. A group of automorphisms of X is called *s-arc-transitive* if it acts transitively on the set of s-arcs of X, and *s-arc-regular* if this action is sharply-transitive, and then the graph X itself is called *s-arc-transitive* or *s-arc-regular* if its full automorphism group $\mathrm{Aut}\,X$ is s-arc-transitive or s-arc-regular, respectively.

If X is cubic (3-valent), then by theorems of Tutte [7, 8], every arc-transitive group of automorphisms of X is s-arc-regular for some $s \le 5$. Moreover, every such group G is a smooth quotient of one of seven finitely-presented groups G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5, which can be presented as follows (see [6, 3]):

$$G_1 = \langle\, h, a \mid h^3 = a^2 = 1 \,\rangle \quad \text{(the modular group)};$$

$$G_2^1 = \langle\, h, p, a \mid h^3 = p^2 = a^2 = 1,\ php = h^{-1},\ a^{-1}pa = p \,\rangle;$$

$$G_2^2 = \langle\, h, p, a \mid h^3 = p^2 = 1,\ a^2 = p,\ php = h^{-1},\ a^{-1}pa = p \,\rangle;$$

$$G_3 = \langle\, h, p, q, a \mid h^3 = p^2 = q^2 = a^2 = 1,\ pq = qp,\ php = h,\ qhq = h^{-1},\ a^{-1}pa = q \,\rangle;$$

$$\begin{aligned} G_4^1 = \langle\, h, p, q, r, a \mid\ & h^3 = p^2 = q^2 = r^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ (qr)^2 = p, \\ & h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle; \end{aligned}$$

$$\begin{aligned} G_4^2 = \langle\, h, p, q, r, a \mid\ & h^3 = p^2 = q^2 = r^2 = 1,\ a^2 = p,\ pq = qp,\ pr = rp,\ (qr)^2 = p, \\ & h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle; \end{aligned}$$

$$\begin{aligned} G_5 = \langle\, h, p, q, r, s, a \mid\ & h^3 = p^2 = q^2 = r^2 = s^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ ps = sp, \\ & qr = rq,\ qs = sq,\ (rs)^2 = pq,\ h^{-1}ph = p,\ h^{-1}qh = r, \\ & h^{-1}rh = pqr,\ shs = h^{-1},\ a^{-1}pa = q,\ a^{-1}ra = s \,\rangle. \end{aligned}$$

In fact if G is s-arc-regular, then G is a smooth quotient of G_s or G_s^i, where $i = 1$

or 2 depending on whether or not the group contains an involution a that reverses an arc (in the cases where s is even). Conversely, every smooth epimorphism from G_s or G_s^i to a finite group G gives rise to a connected cubic graph on which G acts as an s-arc-regular group of automorphisms.

In this paper, we determine all arc-transitive abelian regular covers of the Heawood graph $\mathcal{H}$, which is the incidence graph of the Fano plane, of order 14. This graph is cubic and 4-arc-regular, with automorphism group $\mathrm{PGL}_2(7)$ of order 336, which is a smooth quotient of the group G_4^1, say G_4^1/N. The Heawood graph also admits eight other arc-transitive groups of automorphisms, lying in a single conjugacy class, with each being isomorphic to a semi-direct product $C_7 \rtimes_3 C_6$ (where the 3 indicates that a generator of C_6 conjugates each element of C_7 to its 3rd power), of order 42. One of these arc-regular groups is G_1/N, which for the time being we may call B.

If Y is an arc-transitive regular cover of $\mathcal{H}$, then some arc-transitive group of automorphisms of Y consists of the lifts of all elements in an arc-transitive subgroup of $\mathrm{Aut}\,\mathcal{H}$, and we may take the latter subgroup to be $B = G_1/N \cong C_7 \rtimes_3 C_6$, and the former group of automorphisms of Y as G_1/J for some normal subgroup J of G_1 contained in N.

We seek all possibilities for J such that the covering group N/J is finite and abelian. In fact, since every finite abelian subgroup is a direct product of its Sylow subgroups, we can restrict our search to those J for which the index $|N\!:\!J|$ is a prime-power. We do this for powers of primes other than 7 in Section 3, and for powers of 7 in Section 4, after some preliminary observations in Section 2. Then we summarise our findings in Section 5, and consider the possibility of additional automorphisms in Section 6, before giving a complete classification as our main theorem in Section 7.

One remarkable finding is that although every arc-transitive group of automorphisms of the Heawood graph $\mathcal{H}$ is either 1-arc-regular or 4-arc-regular, there exist regular covers of $\mathcal{H}$ that are 2-arc-regular. Also in some cases, the covering graph can be obtained in two different ways, via non-isomorphic groups of covering transformations (having the same order but different exponent).

2 First steps

Take the group G_4^1, with presentation $\langle\, h, a, p, q, r \mid h^3 = a^2 = p^2 = q^2 = r^2 = (pq)^2 = (pr)^2 = p(qr)^2 = h^{-1}phq = h^{-1}qhpq = (hr)^2 = (ap)^2 = aqar = 1 \,\rangle$, and observe that the three elements h, a and p suffice as generators (because $q = h^{-1}ph$ and $r = aqa$). This group G_4^1 has two normal subgroups of index 336, both with quotient $\mathrm{PGL}(2, 7)$, but these are interchanged by the outer automorphism (induced by conjugation by an element of the larger group G_5) that takes the three generators h, a and p to h, ap and p respectively, and so without loss of generality we can take either one of them.

We will take the one that is contained in the subgroup $G_1 = \langle h, a\rangle$; this is a normal subgroup N of index 42 in G_1 with $G_1/N = B \cong C_7 \rtimes_3 C_6$.

Using Reidemeister-Schreier theory (or the `Rewrite` command in MAGMA [1]), we find that the subgroup N is free of rank 8, on generators

$$
\begin{array}{ll}
w_1 = (ha)^6, & w_2 = hah^{-1}ah^{-1}ahahah^{-1}a, \\
w_3 = (h^{-1}a)^6, & w_4 = h^{-1}ahah^{-1}ah^{-1}ahaha, \\
w_5 = hah^{-1}ahah^{-1}ah^{-1}aha, & w_6 = hahah^{-1}ahah^{-1}ah^{-1}a, \\
w_7 = h^{-1}ahahah^{-1}ahah^{-1}a, & w_8 = h^{-1}ah^{-1}ahahah^{-1}aha.
\end{array}
$$

Easy calculations show that the generators h, a and p act by conjugation particularly nicely, as below:

$$
\begin{array}{rclrclrcl}
h^{-1}w_1h &=& w_3^{-1} & a^{-1}w_1a &=& w_3^{-1} & p^{-1}w_1p &=& w_5 \\
h^{-1}w_2h &=& w_4^{-1} & a^{-1}w_2a &=& w_2^{-1} & p^{-1}w_2p &=& w_6 \\
h^{-1}w_3h &=& w_2w_7^{-1} & a^{-1}w_3a &=& w_1^{-1} & p^{-1}w_3p &=& w_7 \\
h^{-1}w_4h &=& w_3^{-1}w_6 & a^{-1}w_4a &=& w_8^{-1} & p^{-1}w_4p &=& w_8 \\
h^{-1}w_5h &=& w_8^{-1} & a^{-1}w_5a &=& w_7^{-1} & p^{-1}w_5p &=& w_1 \\
h^{-1}w_6h &=& w_7^{-1} & a^{-1}w_6a &=& w_6^{-1} & p^{-1}w_6p &=& w_2 \\
h^{-1}w_7h &=& w_1w_4^{-1} & a^{-1}w_7a &=& w_5^{-1} & p^{-1}w_7p &=& w_3 \\
h^{-1}w_8h &=& w_5w_8^{-1} & a^{-1}w_8a &=& w_4^{-1} & p^{-1}w_8p &=& w_4.
\end{array}
$$

Now take the quotient G_4^1/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^8$ by the group $G_4^1/N \cong \mathrm{PGL}(2,7)$, and replace the generators h, a, p and all w_i by their images in this group. Also let K denote the subgroup N/N', and let G be G_1/N'. Then, in particular, G is an extension of $\mathbb{Z}^8$ by $B = G_1/N \cong C_7 \rtimes_3 C_6$.

By the above observations, we see that the generators h, a and p induce linear transformations of the free abelian group $K \cong \mathbb{Z}^8$ as follows:

$$
h \mapsto \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
\end{pmatrix},
$$

$$
a \mapsto \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

and

$$p \mapsto \begin{pmatrix} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \end{pmatrix}.$$

These matrices generate a group isomorphic to $\mathrm{PGL}(2,7)$, with the first two generating a subgroup isomorphic to $C_7 \rtimes_3 C_6$.

Next, the character table of the group $C_7 \rtimes_3 C_6$ is as follows:

Element order	1	2	3	3	6	6	7
Class size	1	7	7	7	7	7	6
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	1	1	λ	λ^2	λ^2	λ	1
χ_4	1	1	λ^2	λ	λ	λ^2	1
χ_5	1	-1	λ	λ^2	$-\lambda^2$	$-\lambda$	1
χ_6	1	-1	λ^2	λ	$-\lambda$	$-\lambda^2$	1
χ_7	6	0	0	0	0	0	-1

where λ is a primitive cube root of 1.

By inspecting traces of the matrices of orders 2, 3, 6 and 7 induced by each of a, $h^{\pm 1}$, $(ah)^{\pm 1}$ and $[a,h]$, we see that the character of the 8-dimensional representation of $C_7 \rtimes_3 C_6$ over $\mathbb{Q}$ associated with the above action of $G = \langle h, a\rangle$ on K is $\chi_5+\chi_6+\chi_7$, which is expressible as the sum of $\chi_5+\chi_6$ and χ_7, the characters of two irreducible representations over $\mathbb{Q}$ of dimensions 2 and 6.

In the next two sections, for every positive integer m we let $K^{(m)}$ denote the subgroup of K generated by the mth powers of all its elements, and if m is a prime-power, say $m = k^\ell$, then we will consider G_1-invariant subgroups of each layer K_{j-1}/K_j of $K/K^{(m)}$, where $K_j = K^{(k^j)}$ for every non-negative integer j, in order to find G_1-invariant subgroups of $K/K^{(m)}$ itself.

3 Characteristic other than 7

When we reduce by any prime k, the quotient $K/K^{(k)} \cong (\mathbb{Z}_k)^8$ is the direct sum of two G_1-invariant subgroups of ranks 2 and 6, and the latter is irreducible when $k \neq 7$. In fact, these two subgroups are the images of the normal subgroups U and V of ranks 2 and 6 in G generated by

$$u_1 = w_1 w_3 w_5^{-1} w_6^{-1} w_7^{-1} \quad \text{and} \quad u_2 = w_2 w_4 w_5^{-1} w_6^{-1} w_7^{-1} w_8$$

and

$$v_1 = w_1,\ v_2 = w_2 w_7^{-1},\ v_3 = w_3,\ v_4 = w_4 w_8^{-1},\ v_5 = w_5 w_7^{-1} \ \text{ and } \ v_6 = w_6 w_7^{-1} w_8;$$

with conjugation of the respective generators by h and a given as follows:

$$u_1^h = u_1^{-1} u_2, \quad u_2^h = u_1^{-1}, \quad u_1^a = u_1^{-1} \quad \text{and} \quad u_2^a = u_2^{-1},$$

$$v_1^h = v_3^{-1}, \quad v_2^h = v_1^{-1}, \quad v_3^h = v_2, \quad v_4^h = v_3^{-1} v_5^{-1} v_6, \quad v_5^h = v_1^{-1} v_4, \quad v_6^h = v_1^{-1} v_4 v_5,$$

$$v_1^a = v_3^{-1}, \quad v_2^a = v_2^{-1} v_5, \quad v_3^a = v_1^{-1}, \quad v_4^a = v_4, \quad v_5^a = v_5, \quad v_6^a = v_4^{-1} v_5 v_6^{-1}.$$

Hence for every prime $k \neq 7$ and every pair (c, d) of integer powers of k, there exists a G_1-invariant subgroup L of K with index $|K:L| = c^2 d^6$ and with quotient $K/L \cong (\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^6$, generated by the elements u_i^c for $1 \le i \le 2$ and v_j^d for $1 \le j \le 6$.

When $k \equiv 2 \bmod 3$ and $k > 2$, the corresponding subgroups of $K/K^{(k)}$ are both irreducible as G_1-invariant subgroups, since the mod k reductions of the characters $\chi_5 + \chi_6$ and χ_7 are irreducible over $\mathbb{Z}_k$. The same holds also when $k = 2$, since there is no G_1-invariant cyclic subgroup of the rank 2 subgroup in that case. Hence for every prime $k \equiv 2 \bmod 3$, the only G_1-invariant subgroups of K with index a power of k are the subgroups with quotients $K/L \cong (\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^6$ described above.

When $k \equiv 1 \bmod 3$, however, the rank 2 subgroup of $K/K^{(k)}$ splits into the direct sum of two G_1-invariant subgroups of rank 1, generated by the images of

$$z_t = w_1 w_2^t w_3 w_4^t w_5^{t^2} w_6^{t^2} w_7^{t^2} w_8^t$$

for $t \in \{\lambda, \lambda^2\}$, where λ is a primitive cube root of 1 in $\mathbb{Z}_k$.

Here

$$z_t^a = w_1^{-1} w_2^{-t} w_3^{-1} w_4^{-t} w_5^{-t^2} w_6^{-t^2} w_7^{-t^2} w_8^{-t} = z_t^{-1},$$

while

$$z_t^h = w_1^{-t^2} w_2^{-1} w_3^{1+t} w_4^{t+t^2} w_5^{-t} w_6^{-t} w_7^{1+t^2} w_8^{t+t^2},$$

the image of which in $K/K^{(k)}$ is $z_t^{-t^2}$, since $t^2 + t + 1 \equiv 0 \bmod k$ in each case. The same holds when k is replaced by a higher power of K, say $m = k^\ell$: if λ is a primitive cube root of 1 in $\mathbb{Z}_m$, and $z_t = w_1 w_2^t w_3 w_4^t w_5^{t^2} w_6^{t^2} w_7^{t^2} w_8^t$ for $t \in \{\lambda, \lambda^2\}$, then the image of each of z_λ and z_{λ^2} generates a G_1-invariant subgroup of rank 1 in $K/K^{(m)}$, and their direct sum is a G_1-invariant subgroup of rank 2. Moreover, the latter is complementary to the image of V (of rank 6) when $k \neq 7$.

It follows that for every prime $k \equiv 1 \bmod 3$, and for every triple (b, c, d) of powers of k with $b \neq c$, there is also a G_1-invariant subgroup L of K with index $|K:L| = bcd^6$ and quotient $K/L \cong \mathbb{Z}_b \oplus \mathbb{Z}_c \oplus (\mathbb{Z}_d)^6$, generated by the images of the elements $(z_\lambda)^b$, $(z_{\lambda^2})^c$ and v_j^d for $1 \le j \le 6$. Moreover, when $k \neq 7$, each layer of any G_1-invariant subgroup L of K with index a power of k must have rank 1, 2, 6, 7 or 8, and it is easy to see that there are no other possibilities for L (when $k \equiv 1 \bmod 3$ and $k \neq 7$).

When $k = 3$, the quotient $K/K^{(k)} \cong (\mathbb{Z}_3)^8$ has six G_1-invariant subgroups. These include the subgroups of ranks 0, 2, 6 and 8 that occur for every other prime k, plus the cyclic subgroup generated by the image of $z_1 = w_1w_2w_3w_4w_5w_6w_7w_8$ (which coincides with the image of $u_1u_2 = w_1w_2w_3w_4w_5^{-2}w_6^{-2}w_7^{-2}w_8$), and the subgroup of rank 7 generated by the images of z_1 (or u_1u_2) and the elements v_j for $1 \le j \le 6$.

In $K/K^{(9)}$, however, there is no G_1-invariant cyclic subgroup of order 9; the only G_1-invariant subgroups of $K/K^{(9)}$ of rank 1 or 2 are unique subgroups isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_9 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_9 \oplus \mathbb{Z}_9$, generated by the images of $\{(u_1u_2)^3\}$, $\{(u_1u_2)^3, u_2^3\}$ (or $\{u_1^3, u_2^3\}$), $\{u_1u_2, u_2^3\}$ and $\{u_1u_2, u_2\}$ (or $\{u_1, u_2\}$), respectively. It follows that every G_1-invariant subgroup L of K with index a power of 3 is generated by the elements $(u_1u_2)^b$, u_2^c and v_j^d for $1 \le j \le 6$, where b, c and d are powers of 3 with $c = b$ or $3b$, in which case $|K\!:\!L| = bcd^6$ and $K/L \cong \mathbb{Z}_b \oplus \mathbb{Z}_c \oplus (\mathbb{Z}_d)^6$.

This completes the analysis of G_1-invariant subgroups of $K/K^{(m)}$ when m is a power of some prime $k \ne 7$. These will be summarised in a table in Section 5.

4 Characteristic 7

The case $k = 7$ is not quite so straightforward. In this case, each layer can have rank 0, 1, 2, 3, 4, 5, 6, 7 or 8, depending on the layers above it. Here, as we will see, in $K/K^{(7)}$ the images of the subgroups U and V of ranks 2 and 6 considered in the previous section intersect non-trivially in a subgroup of rank 1.

To describe the possibilities for a G_1-invariant subgroup of each layer, again it helps to let λ be a primitive cube root of 1 in $\mathbb{Z}_m$ (when $m = 7^\ell$), and define $z_t = w_1w_2^tw_3w_4^tw_5^{t^2}w_6^{t^2}w_7^{t^2}w_8^t$ for $t \in \{\lambda, \lambda^2\}$. This time, however, we choose λ so that $\lambda \equiv 2 \bmod 7$ (while $\lambda^2 \equiv 4 \bmod 7$). Also, take $v_1 = w_1$, $v_2 = w_2w_7^{-1}$, $v_3 = w_3$, $v_4 = w_4w_8^{-1}$, $v_5 = w_5w_7^{-1}$ and $v_6 = w_6w_7^{-1}w_8$ (as before), and define $y_t = w_7w_8^t$ for each $t \in \{\lambda, \lambda^2\}$. Then an alternative basis for the group $K/K^{(m)}$ is formed by the images of the following eight elements:

$$\begin{aligned}
&x_1 = z_\lambda = w_1w_2^\lambda w_3w_4^\lambda w_5^{\lambda^2}w_6^{\lambda^2}w_7^{\lambda^2}w_8^\lambda, && x_2 = z_{\lambda^2} = w_1w_2^{\lambda^2}w_3w_4^{\lambda^2}w_5^\lambda w_6^\lambda w_7^\lambda w_8^{\lambda^2},\\
&x_3 = v_2v_3v_4^2v_5v_6^2 = w_2w_3w_4^2w_5w_6^2w_7^{-4}, && x_4 = v_3v_6^{-2} = w_3w_6^{-2}w_7^2w_8^{-2},\\
&x_5 = v_4v_5^3 = w_4w_5^3w_7^{-3}w_8^{-1}, && x_6 = v_6y_\lambda^2 = w_6w_7w_8^{1+2\lambda},\\
&x_7 = v_6 = w_6w_7^{-1}w_8, && x_8 = y_{\lambda^2} = w_7w_8^{\lambda^2}.
\end{aligned}$$

With help from Magma [1], we find that the group $K/K^{(7)}$ of order 7^8 has exactly 22 G_1-invariant subgroups. We will denote the trivial subgroup by T_0 and the group $K/K^{(7)}$ itself by T_{21}, and then the 20 non-trivial proper G_1-invariant subgroups can be labelled T_1 to T_{20}, and summarised as follows (in Table 4.1):

	Rank	Generated by images of		Rank	Generated by images of
T_1	1	x_1	T_2	1	x_2
T_3	2	$x_1,\ x_2$	T_4	2	$x_1,\ x_3$
T_5	3	$x_1,\ x_2,\ x_3$	T_6	3	$x_1,\ x_3,\ x_4$
T_7	4	$x_1,\ x_2,\ x_3,\ x_4$	T_8	4	$x_1,\ x_3,\ x_4,\ x_5$
T_9	5	$x_1,\ x_2,\ x_3,\ x_4,\ x_5$	T_{10}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_6$
T_{11}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2x_6$	T_{12}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^2x_6$
T_{13}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^3x_6$	T_{14}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^4x_6$
T_{15}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^5x_6$	T_{16}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^6x_6$
T_{17}	6	$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6$	T_{18}	6	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^4x_6,\ x_7$
T_{19}	7	$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7$	T_{20}	7	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^4x_6,\ x_7,\ x_8$

Table 4.1: The non-trivial proper G_1-invariant subgroups of $K/K^{(7)}$

When the exponent m of K/L is a higher power of 7, say $m = 7^\ell$ with $\ell > 1$, finding the G_1-invariant subgroups of $K/K^{(m)}$ is much more challenging than in earlier cases (namely in the previous section and in [4]).

For all $j > 0$, the G_1-invariant subgroups of the jth layer $K_{j-1}/K_j = K^{(7^{j-1})}/K^{(7^j)}$ of K are isomorphic to the G_1-invariant subgroups of $K/K^{(7)}$, and are generated by the images of the (7^{j-1})th powers of the corresponding sets of x_i in each case. In some sense, what we have to do is see how the possibilities at each layer can fit together.

For each $t \in \{\lambda, \lambda^2\}$ the image of z_t generates a G_1-invariant subgroup of rank 1 in $K/K^{(m)}$, and these two subgroups may be viewed as a tower of copies of T_1 and a tower of copies of T_2 (from Table 4.1). The images of z_λ and z_{λ^2} together generate a G_1-invariant subgroup of rank 2, coinciding with the image of the subgroup U defined earlier, since in $K/K^{(m)}$ the image of z_t is the same as the image of $u_1u_2^t$ for each t (because $-1-t \equiv t^2 \bmod m$). (Also conversely, $z_\lambda^{\lambda}z_{\lambda^2}^{-1} = u_1^{\lambda-1}$ and $z_\lambda^{-1}z_{\lambda^2} = u_2^{\lambda^2-\lambda}$.) This subgroup is a tower of copies of the subgroup T_3 from Table 4.1.

Also, and again as before, the subgroup V generated by $v_1 = w_1,\ v_2 = w_2w_7^{-1}$, $v_3 = w_3,\ v_4 = w_4w_8^{-1},\ v_5 = w_5w_7^{-1}$ and $v_6 = w_6w_7^{-1}w_8$ is G_1-invariant, and so this gives a G_1-invariant homocyclic subgroup of rank 6 in $K/K^{(m)}$, which can be viewed as a tower of copies of T_{18}. (It is an easy exercise to show by arithmetic mod 7 that in $K/K^{(7)}$, the images of each of the generators $x_1, x_3, x_4, x_5, x_2^4x_6$ and x_7 (for T_{18}) is expressible in terms of the images of the generators v_1 to v_6 of V.)

Note that the intersection of the images of the rank 6 subgroup V and the rank 2 subgroup U (or equivalently, the intersection of the T_3 and T_{18} towers) is neither trivial nor one of the rank 1 towers generated by z_λ and z_{λ^2}, except in the case $m = 7$: in fact, it is the cyclic subgroup of order 7 generated by the image of $z_\lambda{}^{\frac{m}{7}}$ $(= x_1{}^{\frac{m}{7}})$.

Next, for each $t \in \{\lambda, \lambda^2\}$, the image of the subgroup generated by $V \cup \{y_t\}$ is a

G_1-invariant subgroup of rank 7 in $K/K^{(m)}$, since

$$y_t^h = w_1 w_4^{-1} w_5^t w_8^{-t} = w_1 (w_4 w_8^{-1})^{-1} (w_5 w_7^{-1})^t (w_7 w_8^t)^t w_8^{-(1+t+t^2)} = v_1 v_4^{-1} v_5^t y_t^t w_8^{-(1+t+t^2)}$$

(with $1+t+t^2 \equiv 0 \bmod m$), while

$$y_t^a = w_5^{-1} w_4^{-t} = (w_4 w_8^{-1})^{-t} (w_5 w_7^{-1})^{-1} (w_7 w_8^t)^{-1} = v_4^{-t} v_5^{-1} y_t^{-1}.$$

These two homocyclic subgroups of rank 7 may be viewed as a tower of copies of T_{19} (when $t = \lambda$) and a tower of copies of T_{20} (when $t = \lambda^2$), since in $K/K^{(7)}$ the images of x_2, x_6 and x_8 coincide with those of $v_1 v_2^4 v_3 v_4^4 v_5^2 v_6^2 y_\lambda^3$, $v_6 y_\lambda^2$ and y_{λ^2} respectively. The only other G_1-invariant subgroup of $T_{21} = K/K^{(7)}$ of rank 6, namely T_{17}, is generated by the images of $v_1 v_6^2$, $v_2 v_6^5$, $v_3 v_6^5$, $v_4 v_6^5$, $v_5 v_6^3$ and $v_6 y_\lambda^2$.

It turns out that the above towers of copies of T_1, T_2, T_3, T_{18}, T_{19} or T_{20} account for all of the homocyclic G_1-invariant subgroups of exponent m in $K/K^{(m)}$, but that will not become clear until we have found all the G_1-invariant subgroups of $K/K^{(m)}$, below.

To see exactly what happens, it is helpful to consider the case $m = 7^2 = 49$. Subgroups of $K/K^{(49)}$ that have rank 8 must all have second layer equal to K_1/K_2 (and a subgroup of $K/K^{(7)}$ as first layer), and are not so interesting for us. Similarly, subgroups of exponent 7 have trivial first layer, and we will ignore those for now.

There are exactly 101 non-trivial subgroups of $K/K^{(49)}$ of exponent 49 and rank at most 7 that are normal in $G/K^{(49)}$, and these can be summarised as follows, with $V^{(j)}$ denoting the set $\{{v_1}^j, {v_2}^j, {v_3}^j, {v_4}^j, {v_5}^j, {v_6}^j\}$ of jth powers of the generators of V:

Rank 1:
- two subgroups isomorphic to $\mathbb{Z}_{49}$, generated by the images of x_1 and x_2;

Rank 2:
- three subgroups isomorphic to $\mathbb{Z}_{49} \oplus \mathbb{Z}_7$, generated by the images of $\{x_1, {x_2}^7\}, \{x_1, {x_3}^7\}$ and $\{x_2, {x_1}^7\}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^2$, generated by the image of $\{x_1, x_2\}$;

Rank 3:
- three subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^2$, generated by the images of $\{x_1, {x_2}^7, {x_3}^7\}$, $\{x_1, {x_3}^7, {x_4}^7\}$ and $\{x_2, {x_1}^7, {x_3}^7\}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^2 \oplus \mathbb{Z}_7$, generated by the image of $\{x_1, x_2, {x_3}^7\}$;

Rank 4:
- three subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^3$, generated by the images of $\{x_1, {x_2}^7, {x_3}^7, {x_4}^7\}$, $\{x_1, {x_3}^7, {x_4}^7, {x_5}^7\}$ and $\{x_2, {x_1}^7, {x_3}^7, {x_4}^7\}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^2 \oplus (\mathbb{Z}_7)^2$, generated by the image of $\{x_1, x_2, {x_3}^7, {x_4}^7\}$;

Rank 5:
- 15 subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^4$, generated by the images of

$\{x_1, {x_2}^7, {x_3}^7, {x_4}^7, {x_5}^7\}$, $\{x_1, {x_3}^7, {x_4}^7, {x_5}^7, ({x_2}^i x_6)^7\}$ for $0 \le i \le 6$, and $\{x_2\, {x_6}^{7i}, {x_1}^7, {x_3}^7, {x_4}^7, {x_5}^7\}$ for $0 \le i \le 6$;

- 7 subgroups isomorphic to $(\mathbb{Z}_{49})^2 \oplus (\mathbb{Z}_7)^3$, generated by the images of $\{x_1, x_2\, {x_6}^{7i}, {x_3}^7, {x_4}^7, {x_5}^7\}$ for $0 \le i \le 6$;

Rank 6:

- 9 subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^5$, generated by the images of $\{x_1, {x_2}^7, {x_3}^7, {x_4}^7, {x_5}^7, {x_6}^7\}$, $\{x_2, {x_1}^7, {x_3}^7, {x_4}^7, {x_5}^7, {x_6}^7\}$, and $\{x_1 {x_8}^{7i}\} \cup V^{(7)}$ for $0 \le i \le 6$;
- two subgroups isomorphic to $(\mathbb{Z}_{49})^2 \oplus (\mathbb{Z}_7)^4$, generated by the images of $\{x_1, x_2, {x_3}^7, {x_4}^7, {x_5}^7, {x_6}^7\}$ and $\{x_1 {x_8}^{14}, x_3\} \cup V^{(7)}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^3 \oplus (\mathbb{Z}_7)^3$, generated by the image of $\{x_1 {x_8}^{14}, x_3, x_4\} \cup V^{(7)}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^4 \oplus (\mathbb{Z}_7)^2$, generated by the image of $\{x_1 {x_8}^{14}, x_3, x_4, x_5\} \cup V^{(7)}$;
- 7 subgroups isomorphic to $(\mathbb{Z}_{49})^5 \oplus \mathbb{Z}_7$, generated by the images of $\{x_1 {x_8}^{14}, x_3, x_4, x_5, ({x_2}^4 x_6) {x_6}^{7i}\} \cup V^{(7)}$ for $0 \le i \le 6$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^6$, generated by the image of $V^{(1)}$;

Rank 7:

- 9 subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^6$, generated by the images of $\{x_1, {x_8}^7\} \cup V^{(7)}$, $\{x_2, (x_6 x_7^{-1})^7\} \cup V^{(7)}$ and $\{x_1 {x_8}^{7i}, (x_6 x_7^{-1})^7\} \cup V^{(7)}$ for $0 \le i \le 6$;
- 9 subgroups isomorphic to $(\mathbb{Z}_{49})^2 \oplus (\mathbb{Z}_7)^5$, generated by the images of $\{x_1 {x_8}^{14}, x_3, (x_6 x_7^{-1})^7\} \cup V^{(7)}$, $\{x_1, x_3, {x_8}^7\} \cup V^{(7)}$, and $\{x_1 {x_8}^{7i}, x_2, (x_6 x_7^{-1})^7\} \cup V^{(7)}$ for $0 \le i \le 6$;
- three subgroups isomorphic to $(\mathbb{Z}_{49})^3 \oplus (\mathbb{Z}_7)^4$, generated by the images of $\{x_1 {x_8}^{14}, x_2, x_3, (x_6 x_7^{-1})^7\} \cup V^{(7)}$, $\{x_1 {x_8}^{14}, x_3, x_4, (x_6 x_7^{-1})^7\} \cup V^{(7)}$, and $\{x_1, x_3, x_4, {x_8}^7\} \cup V^{(7)}$;
- three subgroups isomorphic to $(\mathbb{Z}_{49})^4 \oplus (\mathbb{Z}_7)^3$, generated by the images of $\{x_1 {x_8}^{14}, x_2, x_3, x_4, (x_6 x_7^{-1})^7\} \cup V^{(7)}$, $\{x_1 {x_8}^{14}, x_3, x_4, x_5, (x_6 x_7^{-1})^7\} \cup V^{(7)}$, and $\{x_1, x_3, x_4, x_5, {x_8}^7\} \cup V^{(7)}$;
- 15 subgroups isomorphic to $(\mathbb{Z}_{49})^5 \oplus (\mathbb{Z}_7)^2$, generated by the images of $\{x_1, x_3, x_4, x_5, ({x_2}^4 x_6) {x_6}^{7i}, {x_8}^7\} \cup V^{(7)}$ for $0 \le i \le 6$, $\{x_1 {x_8}^{14}, x_2, x_3, x_4, x_5\} \cup V^{(7)}$, and $\{x_1 {x_8}^{14}, x_3, x_4, x_5, {x_2}^i x_6, (x_6 x_7^{-1})^7\} \cup V^{(7)}$ for $0 \le i \le 6$;
- three subgroups isomorphic to $(\mathbb{Z}_{49})^6 \oplus \mathbb{Z}_7$, generated by the images of $\{x_1 {x_8}^{14}, x_2, x_3, x_4, x_5, x_6\} \cup V^{(7)}$, $V^{(1)} \cup \{{x_6}^7\}$ and $V^{(1)} \cup \{{x_8}^7\}$;
- two subgroups isomorphic to $(\mathbb{Z}_{49})^7$, generated by the images of $V^{(1)} \cup \{x_6\}$ and $V^{(1)} \cup \{x_8\}$.

Now just as we did for the examples considered in [4], we may represent each of the above subgroups as a pair $(T_i,\ T_j)$ indicating the first layer L_0/L_1 and second layer L_1/L_2 of the subgroup L, respectively, where $L_j = L \cap K_j = L \cap K^{(7^j)}$ for all j. In order, the pairs that occur are as follows:

Rank 1: (T_1, T_1) and (T_2, T_2) once each;

Rank 2: (T_1, T_3), (T_1, T_4) and (T_2, T_3) once each; (T_3, T_3) once;

Rank 3: (T_1, T_5), (T_1, T_6) and (T_2, T_5) once each; (T_3, T_5) once;

Rank 4: (T_1, T_7), (T_1, T_8) and (T_2, T_7) once each; (T_3, T_7) once;

Rank 5: (T_1, T_j) for $9 \leq j \leq 16$ once each, and (T_2, T_9) seven times; (T_3, T_9) seven times;

Rank 6: (T_1, T_{17}) and (T_2, T_{17}) once each, and (T_1, T_{18}) seven times; (T_3, T_{17}) and (T_4, T_{18}) once each; (T_6, T_{18}) once; (T_8, T_{18}) once; (T_{14}, T_{18}) seven times; (T_{18}, T_{18}) once;

Rank 7: (T_1, T_{20}) and (T_2, T_{19}) once each, and (T_1, T_{19}) seven times; (T_4, T_{19}) and (T_4, T_{20}) once each, and (T_3, T_{19}) seven times; (T_5, T_{19}), (T_6, T_{19}) and (T_6, T_{20}) once each; (T_7, T_{19}), (T_8, T_{19}) and (T_8, T_{20}) once each; (T_{14}, T_{20}) seven times, and (T_j, T_{19}) for $9 \leq j \leq 16$ once each; (T_{17}, T_{19}), (T_{18}, T_{19}) and (T_{18}, T_{20}) once each; (T_{19}, T_{19}) and (T_{20}, T_{20}) once each.

Note that these pairs are also precisely the pairs that can occur for the G_1-invariant subgroups of any given 'double-layer' section K_{j-1}/K_{j+1} of K.

One thing that is immediately clear from them is that each allowable pair occurs either once only, or exactly seven times. Those that occur seven times are the following: (T_1, T_{18}), (T_1, T_{19}), (T_2, T_9), (T_3, T_9), (T_3, T_{19}), (T_{14}, T_{18}) and (T_{14}, T_{20}); these are the cases involving an extra parameter i, with $0 \leq i \leq 6$.

Moreover, the generating sets for the subgroups that arise in the case of the pair (T_3, T_9) are easily obtained from those for the pair (T_2, T_9), simply by adjoining $x_1 = z_\lambda$, the generator of a rank 1 tower. Similarly, those for the pair (T_1, T_{19}) are easily obtained from those for the pair (T_1, T_{18}), by adjoining $(x_6 x_7^{-1})^7 = y_\lambda^7$, while those for the pair (T_3, T_{19}) can be obtained from those for the pair (T_1, T_{19}) by adjoining $x_2 = z_{\lambda^2}$ (or from the pair (T_2, T_9) by adjoining z_{λ^2} and y_λ^7), and those for the pair (T_{14}, T_{20}) can be obtained from those for the pair (T_{14}, T_{18}) by adjoining ${x_8}^7 = {y_{\lambda^2}}^7$. Adjoining these extra generators does not create any particular complications, and so for larger values of m, we need only pay close attention to the cases involving the pairs (T_1, T_{18}), (T_2, T_9) and (T_{14}, T_{18}).

When $m = 343$, there are precisely 216 G_1-invariant subgroups of $K/K^{(m)}$ that have exponent m and rank at most 7, and we find the following triples occur for the subgroups in the first three layers of these subgroups:

Rank 1: (T_1, T_1, T_1) and (T_2, T_2, T_2) once each;

Rank 2: (T_1, T_1, T_3), (T_2, T_2, T_3), (T_1, T_3, T_3), (T_2, T_3, T_3), (T_3, T_3, T_3) and (T_1, T_1, T_4) once each;

Rank 3: (T_1, T_1, T_5), (T_2, T_2, T_5), (T_1, T_3, T_5), (T_2, T_3, T_5), (T_3, T_3, T_5) and (T_1, T_1, T_6) once each;

Rank 4: (T_1, T_1, T_7), (T_2, T_2, T_7), (T_1, T_3, T_7), (T_2, T_3, T_7), (T_3, T_3, T_7) and (T_1, T_1, T_8) once each;

Rank 5: (T_1, T_1, T_j) for $9 \leq j \leq 16$ once each,
and (T_2, T_2, T_9), (T_1, T_3, T_9), (T_2, T_3, T_9) and (T_3, T_3, T_9) seven times each;

Rank 6: (T_1, T_1, T_{17}), (T_2, T_2, T_{17}), (T_1, T_3, T_{17}), (T_2, T_3, T_{17}), (T_3, T_3, T_{17}), (T_4, T_{18}, T_{18}), (T_6, T_{18}, T_{18}), (T_8, T_{18}, T_{18}) and (T_{18}, T_{18}, T_{18}) once each,
and (T_1, T_1, T_{18}), (T_1, T_{18}, T_{18}), (T_{14}, T_{18}, T_{18}) seven times each;

Rank 7: (T_2, T_2, T_{19}), (T_2, T_5, T_{19}), (T_2, T_7, T_{19}), (T_2, T_{17}, T_{19}), (T_4, T_{18}, T_{19}), (T_6, T_{18}, T_{19}), (T_8, T_{18}, T_{19}), (T_{18}, T_{18}, T_{19}), (T_2, T_{19}, T_{19}), (T_j, T_{19}, T_{19}) for $4 \leq j \leq 19$, (T_1, T_1, T_{20}), (T_1, T_4, T_{20}), (T_1, T_6, T_{20}), (T_1, T_8, T_{20}),(T_4, T_{18}, T_{20}), (T_6, T_{18}, T_{20}), (T_8, T_{18}, T_{20}), (T_{18}, T_{18}, T_{20}), (T_1, T_{20}, T_{20}), (T_4, T_{20}, T_{20}), (T_6, T_{20}, T_{20}), (T_8, T_{20}, T_{20}), (T_{18}, T_{20}, T_{20}), (T_{20}, T_{20}, T_{20}) once each;
and (T_1, T_1, T_{19}), (T_1, T_3, T_{19}), (T_2, T_3, T_{19}), (T_3, T_3, T_{19}), (T_2, T_9, T_{19}), (T_1, T_{18}, T_{19}), (T_{14}, T_{18}, T_{19}), (T_1, T_{19}, T_{19}), (T_3, T_{19}, T_{19}), (T_1, T_{14}, T_{20}), (T_1, T_{18}, T_{20}), (T_{14}, T_{18}, T_{20}) and (T_{14}, T_{20}, T_{20}), seven times each.

Note that some of these contain successive copies of the same subgroup T_j. In fact it is easy to see that when $\ell > 3$ (and m is divisible by 2401), some subgroups can be made up of layers that include multiple copies of two or more of the T_j; for example, when $0 < u < v < \ell$, the G_1-invariant subgroup of $K/K^{(m)}$ generated by the images of z_λ, $(z_{\lambda^2})^{7^u}$ and $\{(w_i)^m : 1 \leq i \leq 8\}$ may be viewed as a tower of u copies of T_1 sitting on top of $\ell - u$ copies of T_3.

Inspection of the generating sets shows, however, that a tower of more than one copy of T_i and a tower of more than one copy of T_j can occur for $i < j$ only when $(T_i, T_j) = (T_1, T_3)$, (T_2, T_3), (T_1, T_{20}), (T_2, T_{19}), (T_{18}, T_{19}) or (T_{18}, T_{20}). For example, we cannot have a tower of two copies of T_1 on top of a tower of copies of T_{18} or T_{19}, or a tower of two copies of T_2 on top of a tower of copies of T_{18} or T_{20}, or a tower of two copies of T_3 on top of a tower of copies of T_{18}, T_{19} or T_{20}.

Similarly, we cannot have a tower of two copies of T_1 on top of a single copy of T_3 on top of a tower of copies of T_{18} or T_{19}, for example. On the other hand, there are some cases where we can have a single copy of another T_t in between (or above or below) towers of copies of T_i and T_j (for $i \neq j$), such as a tower of copies of T_1 on top of a single copy of T_{18} on top of a tower of copies of T_{19}.

Finally, it is not difficult to see that there is no G_1-invariant subgroup of $K/K^{(m)}$ for $m = 2401$ (and hence for any higher power of 7) which has four distinct layers of rank 1 to 7; in other words, when $m = 7^\ell$ for $\ell > 3$, the quotient $K/K^{(m)}$ must always have a layer of rank 0 or 8, or a repeated layer.

These observations allow us to find all possibilities for a G_1-invariant subgroup of K of 7-power index, classified according to their layer sequences. The results are given in Table 5.1 in the next Section, and can be confirmed to some extent with the help of Magma.

5 Summary

Putting the results of Sections 3 and 4 together, we find that the only possibilities for a normal subgroup L of G contained in K with index $|K:L|$ being a power of a prime k are those included in the summary table below (Table 5.1).

Each row of this table describes a class of such subgroups, and for ease of reference, the jth class is denoted in the left-most column by the symbol of the form 'j_S' where S is a single parameter or sequence of parameters, sometimes with an asterisk added. The parameters b, c and d are powers of k, and unless otherwise indicated, we will take $b = k^t$, $c = k^u$, $d = k^v$, and $e = k^w$. If the asterisk appears, then there are exactly seven subgroups of that type with the given parameters, while if it does not, then there is just one such subgroup. The second column gives conditions on the prime k and the other parameters. The third column gives a description of the subgroup(s) in the class; when $k \neq 7$ this is an explicit generating set for L, but when $k = 7$, we indicate the layers of L from the top down, by a sequence of T_j's (for various j) followed by a term K_w (for $K^{(7^w)} = K^{(e)}$), where $e = 7^w$ is the exponent of K/L. Again we use $V^{(j)}$ to denote the set $\{{v_1}^j, {v_2}^j, {v_3}^j, {v_4}^j, {v_5}^j, {v_6}^j\}$ of jth powers of the generators of V. Finally, the fourth column gives the structure of the quotient K/L.

For notational convenience, we use the symbol ${}^fT_j$ to indicate a subsequence $T_j, \overset{f}{\ldots}, T_j$ of f successive copies of the subgroup T_j. Hence, for example, the sequence $({}^2T_2, T_{19}, K_3)$ denotes a subgroup L such that K/L has exponent $7^3 = 343$, with $L_3 = K_3 = K^{(7^3)}$, and for this subgroup, L/L_3 is a a copy of T_{19} extended by a tower of two copies of T_2 (as in the first of the rank 7 subgroups listed for the case $m = 343$ in the previous section). Since T_2 and T_{19} have ranks 1 and 7, for this example we have quotient $K/L \cong (\mathbb{Z}_{343/343})^1 \oplus (\mathbb{Z}_{343/7})^6 \oplus \mathbb{Z}_{343/1} \cong (\mathbb{Z}_{49})^6 \oplus \mathbb{Z}_{343}$.

Explicit generating sets for all these cases can be found in the second author's PhD thesis.

Type	Conditions	Description of L	Quotient K/L
$1_{(c,d)}$	$k \neq 7$	$\langle u_1^c, u_2^c, V^{(d)} \rangle$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^6$
$2_{(b,c,d)}$	$k \equiv 1 \bmod 3$; $k \neq 7$; $b \neq c$	$\langle z_\lambda^b, z_{\lambda^2}^c, V^{(e)} \rangle$	$\mathbb{Z}_b \oplus \mathbb{Z}_c \oplus (\mathbb{Z}_d)^6$
$3_{(c,d)}$	$k = 3$	$\langle (u_1u_2)^c, u_2^{3c}, V^{(d)} \rangle$	$\mathbb{Z}_c \oplus \mathbb{Z}_{3c} \oplus (\mathbb{Z}_d)^6$
4_e	$k = 7$	$({}^wT_0, K_w)$	$(\mathbb{Z}_e)^8$
$5_{(d,e)}$	$k = 7$; $d < e$	$({}^vT_0, {}^{w-v}T_1, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_e)^7$
$6_{(d,e)}$	$k = 7$; $d < e$	$({}^vT_0, {}^{w-v}T_2, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_e)^7$
$7_{(c,d,e)}$	$k = 7$; $c < d < e$	$({}^uT_0, {}^{v-u}T_1, {}^{w-v}T_3, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_e)^6$
$8_{(c,d,e)}$	$k = 7$; $c < d < e$	$({}^uT_0, {}^{v-u}T_2, {}^{w-v}T_3, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_e)^6$
$9_{(d,e)}$	$k = 7$; $d < \frac{e}{7}$	$({}^vT_0, {}^{w-v-1}T_1, T_4, K_w)$	$\mathbb{Z}_d \oplus \mathbb{Z}_{\frac{e}{7}} \oplus (\mathbb{Z}_e)^6$

$10_{(d,e)}$	$k=7;\ d<e$	$({}^{v}T_0, {}^{w-v}T_3, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_e)^6$
11_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_4, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^6$
$12_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_5, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{e}{7}} \oplus (\mathbb{Z}_e)^5$
$13_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_5, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{e}{7}} \oplus (\mathbb{Z}_e)^5$
$14_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_5, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^5$
$15_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_2, T_5, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^5$
$16_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_6, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^5$
$17_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_5, K_w)$	$(\mathbb{Z}_d)^2 \oplus \mathbb{Z}_{\frac{e}{7}} \oplus (\mathbb{Z}_e)^5$
18_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_5, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^5$
19_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_6, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^5$
$20_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_7, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^4$
$21_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_7, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^4$
$22_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_7, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^4$
$23_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_2, T_7, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^4$
$24_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_8, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^4$
$25_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_7, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^4$
26_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_7, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^4$
27_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_8, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^4$
${28_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_9, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^3$
${29_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_9, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^3$
$30_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_9, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$31_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{10}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$32_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{11}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$33_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{12}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$34_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{13}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$35_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{14}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$36_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{15}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$37_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{16}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
${38_{(d,e)}}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_2, T_9, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
${39_{(d,e)}}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_9, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^3$
40_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_9, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$

41_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{10}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
42_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{11}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
43_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{12}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
44_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{13}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
45_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{14}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
46_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{15}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
47_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{16}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
$48_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_{17}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^2$
$49_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_{17}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^2$
$50_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{17}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^2$
$51_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_2, T_{17}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^2$
${52_{(d,e)}}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{18}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^2$
${53_{(d,e)}}^*$	$k=7;\ d<\frac{e}{49}$	$({}^{v}T_0, T_1, {}^{w-v-1}T_{18}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus (\mathbb{Z}_e)^2$
$54_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_{17}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^2$
$55_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_4, {}^{w-v-1}T_{18}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus (\mathbb{Z}_e)^2$
$56_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_6, {}^{w-v-1}T_{18}, K_w)$	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus (\mathbb{Z}_e)^2$
$57_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_8, {}^{w-v-1}T_{18}, K_w)$	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus (\mathbb{Z}_e)^2$
${58_{(d,e)}}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{14}, {}^{w-v-1}T_{18}, K_w)$	$(\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{7d} \oplus (\mathbb{Z}_e)^2$
59_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{17}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^6 \oplus (\mathbb{Z}_e)^2$
$60_{(d,e)}$	$k=7;\ d<e$	$({}^{v}T_0, {}^{w-v}T_{18}, K_w)$	$(\mathbb{Z}_d)^6 \oplus (\mathbb{Z}_e)^2$
${61_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_{19}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus \mathbb{Z}_e$
${62_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_{19}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus \mathbb{Z}_e$
$63_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_4, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
${64_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_3, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$65_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_6, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$66_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_5, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$67_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_8, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$68_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_7, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
${69_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_{14}, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
${70_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_9, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
${71_{(c,d,e)}}^*$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_1, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_{7c})^5 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
${72_{(c,d,e)}}^*$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_1, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_{7c})^5 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$

$73_{(c,d,e)}{}^*$	$k=7;\ 7c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_{18}, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_e$
$74_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_{17}, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_e$
$75_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{19}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^6 \oplus \mathbb{Z}_e$
$76_{(c,d,e)}$	$k=7;\ c<d<e$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^6 \oplus \mathbb{Z}_e$
$77_{(c,d,e)}$	$k=7;\ c<d<e$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^6 \oplus \mathbb{Z}_e$
$78_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{49}$	$({}^{v}T_0, T_1, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^6 \oplus \mathbb{Z}_e$
$79_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_{19}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus \mathbb{Z}_e$
$80_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{49}$	$({}^{v}T_0, T_3, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$81_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_4, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$82_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_4, {}^{w-v-1}T_{20}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$83_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_4, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_{7c})^4 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$84_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_4, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_{7c})^4 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$85_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_5, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$86_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_6, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$87_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_6, {}^{w-v-1}T_{20}, K_w)$	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$88_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_6, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_{7c})^3 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$89_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_6, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_{7c})^3 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$90_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_7, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$91_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_8, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$92_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_8, {}^{w-v-1}T_{20}, K_w)$	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$93_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_8, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^4 \oplus (\mathbb{Z}_{7c})^2 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$94_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_8, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^4 \oplus (\mathbb{Z}_{7c})^2 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$95_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_9, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$96_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{10}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$97_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{11}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$98_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{12}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$99_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{13}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$100_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{14}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$101_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{15}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$102_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{16}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$103_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{14}, {}^{w-v-1}T_{20}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$

$104_{(c,d,e)}{}^*$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_{14}, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^5 \oplus \mathbb{Z}_{7c} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$105_{(c,d,e)}{}^*$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_{14}, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^5 \oplus \mathbb{Z}_{7c} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$106_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{17}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^6 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_e$
$107_{(c,d,e)}$	$k=7;\ c<d<e$	$({}^{u}T_0, {}^{v-u}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^6 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$108_{(c,d,e)}$	$k=7;\ c<d<e$	$({}^{u}T_0, {}^{v-u}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^6 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$109_{(d,e)}$	$k=7;\ d<e$	$({}^{v}T_0, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_d)^7 \oplus \mathbb{Z}_e$
$110_{(d,e)}$	$k=7;\ d<e$	$({}^{v}T_0, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_d)^7 \oplus \mathbb{Z}_e$

Table 5.1: Possibilities for G_1-invariant subgroup L of K when $G_1/K \cong C_7 \rtimes_3 C_6$
[Note: $b = k^t,\ c = k^u,\ d = k^v$ and $e = k^w$ (with $t, u, v, w \geq 0$) in all relevant cases]

6 Additional automorphisms

In this section, we find out which of the regular covers obtainable from G_1-invariant subgroups of finite prime-power index in $K = N/N'$ admit a larger group of automorphisms than the lift of the group $G_1/N \cong C_7 \rtimes_3 C_6$.

First, we note that none of these regular covers can be 5-arc-transitive, since the Heawood graph itself is not 5-arc-transitive (and in particular, the subgroup N is not normal in the group G_5).

The next possibility we check is that the cover is 4-arc-transitive. To do this, we consider whether or not the G_1-invariant subgroup L is G_4^1-invariant, which we can do by checking whether L is normalised by the additional generator p of G_4^1. If it is, then each layer of L must also be normalised by p, since the subgroups $K_j = K^{k^j}$ of K are characteristic in K. For this reason, we begin by determining which of the G_1-invariant subgroups of $K/K^{(k)}$ are normalised by p. Recall that p conjugates w_i to w_j whenever $j \equiv i + 4 \bmod 8$.

Now for every prime k, it is easy to see that the rank 2 subgroup U generated by $u_1 = w_1 w_3 w_5^{-1} w_6^{-1} w_7^{-1}$ and $u_2 = w_2 w_4 w_5^{-1} w_6^{-1} w_7^{-1} w_8$ is not G_4^1-invariant, since $u_1{}^p = w_1^{-1} w_2^{-1} w_3^{-1} w_5 w_7$, which does not lie in U. Also the rank 6 subgroup V generated by $v_1 = w_1$, $v_2 = w_2 w_7^{-1}$, $v_3 = w_3$, $v_4 = w_4 w_8^{-1}$, $v_5 = w_5 w_7^{-1}$ and $v_6 = w_6 w_7^{-1} w_8$ is not G_4^1-invariant, since $v_1{}^p = w_5$, which does not lie in V. Similarly, when $k \equiv 1 \bmod 3$ and t is a primitive cube root of 1 mod k, the rank 1 subgroup of $K/K^{(k)}$ generated by $z_t = w_1 w_2^t w_3 w_4^t w_5^{t^2} w_6^{t^2} w_7^{t^2} w_8^t$ is not G_4^1-invariant, because $z_t{}^p$ is not expressible as a power of z_t. On the other hand, when $k = 3$, the rank 1 subgroup generated by $z_1 = w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8$ (or by $u_1 u_2$) is G_4^1-invariant, since z_1 is centralized by p. But this does not extend to a rank 1 subgroup of $K/K^{(m)}$ when m is a higher power of 3, since $K/K^{(9)}$ has no cyclic G_1-invariant subgroup of order greater than 3.

It follows that for $k \neq 7$, the only G_1-invariant subgroups of k-power index in K that are also G_4^1-invariant are the rank 8 subgroups $K^{(m)}$ themselves, with covering group $K/K^{(m)} \cong (\mathbb{Z}_m)^8$, for $m = k^\ell$ (for any such k), and the subgroups generated by the images of $(u_1u_2)^{\frac{m}{3}}$ and all w_i^m, with covering group $\mathbb{Z}_{\frac{m}{3}} \oplus (\mathbb{Z}_m)^7$, when $m = 3^\ell$ for some $\ell > 0$. These are the subgroups of types $1_{(m,m)}$ and $3_{(\frac{m}{3},m)}$ in Table 5.1.

In the case $k = 7$, again we let λ be a primitive cube root of 1 mod m, where $m = 7^\ell$ is the exponent of the covering group K/L, chosen such that $\lambda \equiv 2$ mod 7 and $\lambda^2 \equiv 4$ mod 7. For notational convenience, we will write $x \simeq y$ when the elements x and y have the same image in the top layer $K/K^{(7)}$ of K, so that (for example) $z_\lambda \simeq w_1w_2^2w_3w_4^2w_5^4w_6^4w_7^4w_8^2$. The effect of conjugation by p on the generators x_1 to x_8 defined in Section 4 can now be given as follows:

$$\begin{array}{lcl}
x_1 \simeq w_1w_2^2w_3w_4^2w_5^4w_6^4w_7^4w_8^2 & \mapsto & w_1^4w_2^4w_3^4w_4^2w_5w_6^2w_7w_8^2 \simeq x_1^2x_2^2x_3^6x_4x_5^6x_6 \\
x_2 \simeq w_1w_2^4w_3w_4^4w_5^2w_6^2w_7^2w_8^4 & \mapsto & w_1^2w_2^2w_3^2w_4^4w_5w_6^4w_7w_8^4 \simeq x_2^2x_3x_4^6x_5x_6^4x_7^6x_8 \\
x_3 \simeq w_2w_3w_4^2w_5w_6^2w_7^3 & \mapsto & w_1w_2^2w_3^3w_6w_7w_8^2 \simeq x_2x_3^5x_4^4x_6^4 \\
x_4 \simeq w_3w_6^5w_7^2w_8^5 & \mapsto & w_2^5w_3^2w_4^5w_7 \simeq x_1^2x_2^5x_3^2x_5^5x_6^6 \\
x_5 \simeq w_4w_5^3w_7^4w_8^6 & \mapsto & w_1^3w_3^4w_4^6w_8 \simeq x_1^3x_3x_5^5 \\
x_6 \simeq w_6w_7w_8^5 & \mapsto & w_2w_3w_4^5 \simeq x_1^5x_2^2x_3^4x_4^4x_6^2x_7^2x_8^5 \\
x_7 \simeq w_6w_7^6w_8 & \mapsto & w_2w_3^6w_4 \simeq x_1^6x_2x_3^6x_5x_6x_7^3x_8^3 \\
x_8 \simeq w_7w_8^4 & \mapsto & w_3w_4^4 \simeq x_1^6x_2x_3^5x_4^3x_5^6x_6x_7^4x_8^2.
\end{array}$$

Note that the images of x_1^p and x_2^p both lie outside the image of the subgroup generated by x_1, x_2, x_3, x_4 and x_5, and so it follows from the definition of the G_1-invariant subgroups of $K/K^{(7)}$ (in Table 4.1) that none of the subgroups T_1 to T_9 of $K/K^{(7)}$ is normalised by p. Similarly, none of the subgroups $T_{10}, T_{11}, T_{13}, T_{14}, T_{15}, T_{16}, T_{18}$ and T_{20} is normalised by p, since each contains the image of x_3 but not the image of x_3^p, and the subgroups T_{17} and T_{19} are not normalised by p, since they contain the image of x_2 but not the image of x_2^p (and contain the image of x_6 but not the image of x_6^p).

On the other hand, the subgroup T_{12} is normalised by p, because

$$\begin{aligned}
x_1^p &\simeq x_1^2x_2^2x_3^6x_4x_5^6x_6 \simeq x_1^2x_3^6x_4x_5^6(x_2^2x_6), \\
x_3^p &\simeq x_2x_3^5x_4^4x_6^4 \simeq x_3^5x_4^4(x_2^2x_6)^4, \\
x_4^p &\simeq x_1^2x_2^5x_3^2x_5^5x_6^6 \simeq x_1^2x_3^2x_5^5(x_2^2x_6)^6, \\
x_5^p &\simeq x_1^3x_3x_5^5, \quad \text{and} \\
(x_2^2x_6)^p &\simeq x_1^5x_2^6x_3^6x_4^2x_5^2x_6^3 \simeq x_1^5x_3^6x_4^2x_5^2(x_2^2x_6)^3.
\end{aligned}$$

Thus T_{12} is the only non-trivial proper G_1-invariant subgroup of $K/K^{(7)}$ normalised by p. Furthermore, since there are no G_1-invariant subgroups of $K/K^{(49)}$ with T_{12} as both layers, this subgroup can occur in at most one layer of L.

Hence we find that the only G_1-invariant subgroups of 7-power index in K that are also G_4^1-invariant are the subgroups $K^{(m)}$, with covering group $K/K^{(m)} \cong (\mathbb{Z}_m)^8$, with $m = 7^\ell$ for $\ell \geq 0$, plus one subgroup with covering group $(\mathbb{Z}_{\frac{m}{7}})^5 \oplus (\mathbb{Z}_m)^3$ where $m = 7^\ell$, for each $\ell > 0$. These are the subgroups of types 4_m and 43_m in Table 5.1.

Next, we consider the possibility that the G_1-invariant subgroup L of K is also G_2^1-invariant. Of course this is not very likely to happen, since the Heawood graph has no 2-arc-regular group of automorphisms (and in particular, the subgroup K of G/N' itself is not G_2^1-invariant), but remarkably, it does happen.

The group G_2^1 can be obtained as an extension of G_1 by adjoining the involutory automorphism θ of G_1 that takes h and a to h^{-1} and $a^{-1}\,(=a)$, respectively. This is like a reflection, and takes $w_1 = (ha)^6$ to $(h^{-1}a)^6 = w_3$, and vice versa, but takes each of $w_2, w_4, w_5, w_6, w_7, w_8$ to an element outside of K. (For example, $w_8^\theta = (h^{-1}ah^{-1}ahahah^{-1}aha)^\theta = hahah^{-1}ah^{-1}ahah^{-1}a = w_6w_3^{-1}h^{-1}aha$, which does not lie in K, for otherwise K would contain $[h,a] = h^{-1}aha$.)

In particular, θ does not preserve K, but takes K to another subgroup of index 42 in G, with intersection $J = K \cap K^\theta$ having index 7 in K (and index 294 in G). In fact $J = K \cap K^\theta$ is generated by the eight elements $v_1 = w_1$, $v_2 = w_2w_7^{-1}$, $v_3 = w_3$, $v_4 = w_4w_8^{-1}$, $v_5 = w_5w_7^{-1}$, $v_6 = w_6w_7^{-1}w_8$, $y_2 = w_7^{-1}w_8^2$ and w_8^7, with:

$$\begin{aligned}
v_1^\theta &= w_1^\theta = ((ha)^6)^\theta = (h^{-1}a)^6 = w_3 = v_3,\\
v_3^\theta &= w_3^\theta = w_1 = v_1,\\
v_2^\theta &= (w_2w_7^{-1})^\theta = (h^{-1}w_3h)^\theta = hw_1h^{-1} = w_7w_2^{-1} = (w_2w_7^{-1})^{-1} = v_2^{-1},\\
v_4^\theta &= (w_4w_8^{-1})^\theta(h^{-1}ah^{-1}w_3hah)^\theta = hahw_1h^{-1}ah^{-1} = w_5w_8^{-1}w_3w_6^{-1}\\
&= w_3(w_5w_7^{-1})(w_6w_7^{-1}w_8)^{-1} = v_3v_5v_6^{-1},\\
v_5^\theta &= (w_5w_7^{-1})^\theta = (hah^{-1}ahah^{-1}ah^{-1}ah^{-1}ah^{-1}ahah^{-1}ah^{-1}ah)^\theta\\
&= h^{-1}ahah^{-1}ahahahahah^{-1}ahahah^{-1} = w_4w_1^{-1}w_6w_2^{-1}\\
&= w_1^{-1}(w_2w_7^{-1})^{-1}(w_4w_8^{-1})(w_6w_7^{-1}w_8) = v_1^{-1}v_2^{-1}v_4v_6,\\
v_6^\theta &= (w_6w_7^{-1}w_8)^\theta = (hahah^{-1}ahahahah^{-1}ahahahah^{-1}aha)^\theta\\
&= h^{-1}ah^{-1}ahah^{-1}ah^{-1}ah^{-1}ahah^{-1}ah^{-1}ah^{-1}ahah^{-1}a\\
&= w_8w_5^{-1}w_1w_4^{-1}w_7 = w_1(w_4w_8^{-1})^{-1}(w_5w_7^{-1})^{-1} = v_1v_4^{-1}v_5^{-1},\\
y_2^\theta &= (w_7^{-1}w_8^2)^\theta\\
&= (h^{-1}ahahah^{-1}ahah^{-1}ah^{-1}ah^{-1}ahahah^{-1}ahah^{-1}ah^{-1}ahahah^{-1}aha)^\theta\\
&= hah^{-1}ah^{-1}ahah^{-1}ahahahah^{-1}ah^{-1}ahah^{-1}ahahah^{-1}ah^{-1}ahah^{-1}a\\
&= w_2w_3^{-1}w_5w_4^{-1}w_7 = (w_2w_7^{-1})\,w_3^{-1}(w_4w_8^{-1})^{-1}(w_5w_7^{-1})(w_7w_8^2)^3\,w_8^{-7}\\
&= v_2v_3^{-1}v_4^{-1}v_5\,y_2^3\,w_8^{-7},\\
(w_7^7)^\theta &= ((h^{-1}ahahah^{-1}ahah^{-1}a)^7)^\theta = (hah^{-1}ah^{-1}ahah^{-1}aha)^7\\
&= w_2w_3^{-1}w_4w_1^{-1}w_5w_7w_3^{-1}w_8w_1^{-1}w_2w_4w_1^{-1}w_6w_3^{-1}w_8\\
&= w_1^{-3}w_2^2w_3^{-3}w_4^2w_5w_6w_7w_8^2\\
&= w_1^{-3}(w_2w_7^{-1})^2w_3^{-3}(w_4w_8^{-1})^2(w_5w_7^{-1})(w_6w_7^{-1}w_8)(w_7w_8^2)^5w_8^{-7}\\
&= v_1^{-3}v_2^2v_3^{-3}v_4^2v_5v_6\,y_2^5\,(w_8^7)^{-1},\\
(w_8^7)^\theta &= ((h^{-1}ah^{-1}ahahah^{-1}aha)^7)^\theta = (hahah^{-1}ah^{-1}ahah^{-1}a)^7\\
&= w_6w_3^{-1}w_7w_2^{-1}w_5w_1^{-1}w_6w_5w_8^{-1}w_7w_1^{-1}w_6w_3^{-1}w_5w_4^{-1}w_7\\
&= w_1^{-2}w_2^{-1}w_3^{-2}w_4^{-1}w_5^3w_6^3w_7^3w_8^{-1}\\
&= w_1^{-2}(w_2w_7^{-1})^{-1}w_3^{-2}(w_4w_8^{-1})^{-1}(w_5w_7^{-1})^3(w_6w_7^{-1}w_8)^3(w_7w_8^2)^8w_8^{-21}\\
&= v_1^{-2}v_2^{-1}v_3^{-2}v_4^{-1}v_5^3v_6^3\,y_2^8\,(w_8^7)^{-3}.
\end{aligned}$$

It follows that $J = K \cap K^\theta$ contains V, and w_i^7 for all i, as well as $y_\lambda = w_7 w_8^\lambda$ (which is the product of $y_2 = w_7 w_8^2$ and a power of w_8^7), whenever m is a power of 7 and λ is a primitive cube root of 1 mod m with $\lambda \equiv 2$ mod 7. On the other hand, J contains neither $u_1 = w_1 w_3 w_5^{-1} w_6^{-1} w_7^{-1}$ nor $u_2 = w_2 w_4 w_5^{-1} w_6^{-1} w_7^{-1} w_8$ (from Section 3), but J does contain each of u_1^7, u_2^7 and $u_1 u_2^2 = y_2$.

It is also easy to see that this subgroup J is G_1-invariant, by checking the images of V, y_2 and w_8^7 under conjugation by h and a. But in fact K^θ itself is G_1-invariant, because $(K^\theta)^h = K^{\theta h} = K^{h^{-1}\theta} = K^\theta$ and $(K^\theta)^a = K^{\theta a} = K^{a\theta} = K^\theta$, and it follows directly from this that $J = K \cap K^\theta$ is G_1-invariant. Similarly, J is θ-invariant, since $(K \cap K^\theta)^\theta = K^\theta \cap K = K \cap K^\theta$.

We may view the 'top layer' of J as a copy of the rank 7 subgroup T_{19} of $K/K^{(7)}$, with every subsequent layer of J being isomorphic to $(\mathbb{Z}_7)^8$ (generated by the images of the appropriate powers of all the w_i).

Now let L be any G_1-invariant subgroup of finite prime-power index in K, such that L^θ lies in K. Then also L^θ is G_1-invariant, by the same argument as used for K^θ a few lines above. Also L^θ lies in K^θ, so lies in $K \cap K^\theta = J$ as well. In particular, the index $|K : L|$ must be a multiple of $|K : J| = 7$. Hence we may restrict our attention to the case of characteristic 7, and the subgroups we found in Section 4.

Next, consider the commutator $c_{ij} = [w_i, w_j] = w_i^{-1} w_j^{-1} w_i w_j$ of any two of the generators w_i and w_j of K. Since these two elements commute in K, and L^θ lies in K, we know that L^θ (trivially) contains c_{ij}, and it follows that L must contain the θ-image c_{ij}^θ, for all such i and j.

These commutators are easily computed. For example,

$$\begin{aligned}
c_{12}^\theta &= (w_1^\theta)^{-1}(w_2^\theta)^{-1} w_1^\theta w_2^\theta \\
&= (ah)^6(ah^{-1}ahahah^{-1}ah^{-1}ah)(h^{-1}a)^6(h^{-1}ahahah^{-1}ah^{-1}aha) \\
&= (ah)^6 ah^{-1}ahahah^{-1}ahah^{-1}ah^{-1}ah^{-1}ah^{-1}ah^{-1}ahahah^{-1}ah^{-1}aha \\
&= (ah)^6(ah^{-1}ahahah^{-1}ahah^{-1})(ah^{-1})^6 hah^{-1}ahah^{-1}ah^{-1}aha \\
&= w_3^{-1}w_5^{-1}w_1^{-1}w_5 = w_1^{-1}w_3^{-1} = v_1^{-1}v_3^{-1}.
\end{aligned}$$

All such θ-images c_{ij}^θ are given below:

$$\begin{aligned}
c_{12}^\theta &= w_1^{-1}w_3^{-1} = v_1^{-1}v_3^{-1}, \\
c_{13}^\theta &= w_3^{-1}w_1^{-1}w_3w_1 = 1, \\
c_{14}^\theta &= w_3^{-1}w_6^{-1}w_1w_2^{-1}w_5w_1^{-1}w_6 = v_2^{-1}v_3^{-1}v_5, \\
c_{15}^\theta &= w_3^{-1}w_2^{-1}w_1^{-1}w_2 = v_1^{-1}v_3^{-1}, \\
c_{16}^\theta &= w_3^{-1}w_4^{-1}w_2w_7^{-1}w_4 = v_2v_3^{-1}, \\
c_{17}^\theta &= w_3^{-1}w_8^{-1}w_3w_6^{-1}w_1^{-1}w_6w_3^{-1}w_8 = v_1^{-1}v_3^{-1}, \\
c_{18}^\theta &= w_3^{-1}w_7^{-1}w_4w_2^{-1}w_5w_4^{-1}w_7 = v_2^{-1}v_3^{-1}, \\
c_{23}^\theta &= w_5^{-1}w_1w_4^{-1}w_7w_6^{-1}w_5w_1 = v_1^2v_4^{-1}v_6^{-1},
\end{aligned}$$

$$
\begin{aligned}
c_{24}^{\theta} &= w_5^{-1}w_1w_7^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8w_1^{-1}w_6 = v_1^{-1}v_3^{-1}v_4v_5^{-1}v_6^2,\\
c_{25}^{\theta} &= w_5^{-1}w_1w_8^{-1}w_4w_1^{-1}w_2 = v_2v_4v_5^{-1},\\
c_{26}^{\theta} &= w_5^{-1}w_1w_4^{-1}w_7w_2^{-1}w_5w_1^{-1}w_2w_7^{-1}w_4 = 1,\\
c_{27}^{\theta} &= w_5^{-1}w_1w_4^{-1}w_3^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8 = v_3^{-2}v_5^{-1}v_6,\\
c_{28}^{\theta} &= w_5^{-1}w_1w_5^{-1}w_2w_7^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8w_4^{-1}w_7 = v_2v_3^{-1}v_5^{-2}v_6,\\
c_{34}^{\theta} &= w_1^{-1}w_6^{-1}w_1w_4^{-1}w_8w_1^{-1}w_6 = v_1^{-1}v_4^{-1},\\
c_{35}^{\theta} &= w_1^{-1}w_2^{-1}w_6w_7^{-1}w_4w_1^{-1}w_2 = v_1^{-2}v_4v_6,\\
c_{36}^{\theta} &= w_1^{-1}w_4^{-1}w_7w_2^{-1}w_5w_8^{-1}w_3w_6^{-1}w_2w_7^{-1}w_4 = v_1^{-1}v_3v_5v_6^{-1},\\
c_{37}^{\theta} &= w_1^{-1}w_8^{-1}w_3w_7^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8 = v_1^{-2}v_4v_6,\\
c_{38}^{\theta} &= w_1^{-1}w_7^{-1}w_8w_4^{-1}w_7 = v_1^{-1}v_4^{-1},\\
c_{45}^{\theta} &= w_6^{-1}w_1w_5^{-1}w_1w_4^{-1}w_7w_1^{-1}w_2 = v_1v_2v_4^{-1}v_5^{-1}v_6^{-1},\\
c_{46}^{\theta} &= w_6^{-1}w_1w_8^{-1}w_6w_3^{-1}w_2w_7^{-1}w_4 = v_1v_2v_3^{-1}v_4,\\
c_{47}^{\theta} &= w_6^{-1}w_1w_8^{-1}w_3w_1^{-1}w_6w_3^{-1}w_8 = 1,\\
c_{48}^{\theta} &= w_6^{-1}w_1w_5^{-1}w_7w_2^{-1}w_5w_4^{-1}w_7 = v_1v_2^{-1}v_4^{-1}v_6^{-1},\\
c_{56}^{\theta} &= w_2^{-1}w_1w_4^{-1}w_7w_2^{-1}w_5w_2w_7^{-1}w_4 = v_1v_2^{-1}v_5,\\
c_{57}^{\theta} &= w_2^{-1}w_1w_4^{-1}w_3^{-1}w_8w_1^{-1}w_6w_3^{-1}w_8 = v_2^{-1}v_3^{-2}v_4^{-1}v_6,\\
c_{58}^{\theta} &= w_2^{-1}w_1w_5^{-1}w_2w_7^{-1}w_4w_1^{-1}w_5w_4^{-1}w_7 = 1,\\
c_{67}^{\theta} &= w_4^{-1}w_7w_2^{-1}w_1w_4^{-1}w_8w_5^{-1}w_2w_7^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8 = v_3^{-1}v_4^{-1}v_5^{-1}v_6,\\
c_{68}^{\theta} &= w_4^{-1}w_7w_2^{-1}w_6^{-1}w_8w_4^{-1}w_7 = v_2^{-1}v_4^{-2}v_6^{-1},\\
c_{78}^{\theta} &= w_8^{-1}w_3w_6^{-1}w_1w_5^{-1}w_2w_3^{-1}w_8w_4^{-1}w_7 = v_1v_2v_4^{-1}v_5^{-1}v_6^{-1}.
\end{aligned}
$$

Note that every element in the list above is expressible in terms of the generators v_1 to v_6 of the rank 6 subgroup V of K. In fact, each of them is expressible as a word in the following 'base' elements: v_1v_3, $v_2v_3^{-1}$, v_1v_4, $v_1v_2^{-1}v_5$, $v_2v_4^2v_6$ and v_4^7, or perhaps better still, the elements v_1v_4, $v_2v_4^{-1}$, $v_3v_4^{-1}$, $v_5v_4^{-2}$, $v_6v_4^3$ and v_4^7.

So now let F be the subgroup generated by the six elements v_1v_4, $v_2v_4^{-1}$, $v_3v_4^{-1}$, $v_5v_4^{-2}$, $v_6v_4^3$ and v_4^7. Then F contains $v_1^7 = (v_1v_4)^7v_4^{-7}$, and similarly contains v_2^7, v_3^7, v_5^7 and v_6^7, so F has index 7 in V, with $K/F \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_7$. Also it is easy to check using the conjugacy details given at the beginning of Section 3 that F is G_1-invariant. Similarly, using the θ-images of the elements v_i, we can see that F is preserved by θ.

The first layer of F is a copy of T_{14}, since the images of the five elements v_1v_4, $v_2v_4^{-1}$, $v_3v_4^{-1}$, $v_5v_4^{-2}$ and $v_6v_4^3$ in $K/K^{(7)}$ generate the same subgroup as $\{x_1, x_3, x_4, x_5, x_2^4x_6\}$, while all subsequent layers are copies of T_{18}. It follows that F is one of the seven subgroups of type $58_{(1,7)}{}^*$ from Table 5.1, and in fact F can be generated by $\{x_1{x_8}^{14}, x_3, x_4, x_5, ({x_2}^4x_6){x_6}^{28}\} \cup V^{(7)}$. (We leave the reader to prove that the subgroup generated by V does not contain $({x_2}^4 x_6){x_6}^{7i}$ when $i \not\equiv 4 \bmod 7$.)

Now once again, let L be any G_1-invariant subgroup of finite 7-power index in K, such that L^θ lies in K. Then we know that L^θ is G_1-invariant, and $F \subseteq L \subseteq J$. It follows that the top layer of L is isomorphic to a subgroup of $K/K^{(7)}$ containing T_{14} and contained in T_{19}, and so must be a copy of one of T_{14}, T_{17}, T_{18} or T_{19}, while every subsequent layer of L contains a copy of T_{18} and hence is a copy of T_{18}, T_{19}, T_{20} or $T_{21} = K/K^{(7)} \cong (\mathbb{Z}_7)^8$ itself.

Conversely, if L is any G_1-invariant subgroup of K such that $F \subseteq L \subseteq J$, then $F = F^\theta \subseteq L^\theta \subseteq J^\theta = J$, and in particular, also L^θ is a G_1-invariant subgroup of K. Moreover, L^θ has the same index in G_1 as L, and hence the same index in K as L.

The relevant subgroup types from Table 5.1 are given in Table 6.1 below, with the asterisks dropped from types ${58_{(1,e)}}^*$, ${103_{(1,e)}}^*$, ${104_{(1,d,e)}}^*$ and ${105_{(1,d,e)}}^*$ since there is just one subgroup containing F in each of those cases.

Type	Conditions	Description of L	Quotient K/L
45_7	$k = 7$	(T_{14}, K_7)	$(\mathbb{Z}_7)^3$
$58_{(1,e)}$	$k = 7;\ e > 7$	$(T_{14}, {}^{w-1}T_{18}, K_w)$	$\mathbb{Z}_7 \oplus (\mathbb{Z}_e)^2$
59_7	$k = 7$	(T_{17}, K_7)	$(\mathbb{Z}_7)^2$
$60_{(1,e)}$	$k = 7;\ e > 1$	$({}^{w}T_{18}, K_w)$	$(\mathbb{Z}_e)^2$
$100_{(1,e)}$	$k = 7;\ e > 7$	$(T_{14}, {}^{w-1}T_{19}, K_w)$	$(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_e$
$103_{(1,e)}$	$k = 7;\ e > 7$	$(T_{14}, {}^{w-1}T_{20}, K_w)$	$(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_e$
$104_{(1,d,e)}$	$k = 7;\ 7 < d < e$	$(T_{14}, {}^{v-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$\mathbb{Z}_7 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$105_{(1,d,e)}$	$k = 7;\ 7 < d < e$	$(T_{14}, {}^{v-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$\mathbb{Z}_7 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$106_{(1,e)}$	$k = 7;\ e > 7$	$(T_{17}, {}^{w-1}T_{19}, K_w)$	$\mathbb{Z}_7 \oplus \mathbb{Z}_e$
$107_{(1,d,e)}$	$k = 7;\ 1 < d < e$	$({}^{v}T_{18}, {}^{w-v}T_{19}, K_w)$	$\mathbb{Z}_d \oplus \mathbb{Z}_e$
$108_{(1,d,e)}$	$k = 7;\ 1 < d < e$	$({}^{v}T_{18}, {}^{w-v}T_{20}, K_w)$	$\mathbb{Z}_d \oplus \mathbb{Z}_e$
$109_{(1,e)}$	$k = 7;\ e > 1$	$({}^{w}T_{19}, K_w)$	$\mathbb{Z}_e$

Table 6.1: Possibilities for G_1-invariant subgroup L of K lying between F and J
[Note: $d = k^v$ and $e = k^w$ in all relevant cases]

Next, the following is helpful in considering the effect of θ on these subgroups.

Proposition 6.1 *Let L be any G_1-invariant subgroup of K such that $F \subseteq L \subseteq J$.*

(a) *If K/L has exponent $m = 7^\ell$ where $\ell > 1$, then the ℓth layer of L^θ contains the image of ${x_6}^{\frac{m}{7}} = (v_6 y_\lambda^2)^{\frac{m}{7}}$, and hence is a copy of T_{19} or T_{21}.*

(b) *If the top two layers of L are copies of T_{14} and T_{18}, then the top two layers of L^θ are copies of T_{14} and T_{19}, or T_{14} and T_{18}, according to whether or not the third layer of L contains a copy of T_{20}.*

(c) *If the top two layers of L are copies of T_{14} and T_{19}, then the top two layers of L^θ are copies of T_{14} and T_{18}, or T_{14} and T_{19}, according to whether the third layer of L has rank 7 or 8.*

(d) *If the top two layers of L are copies of T_{14} and T_{20}, then the top two layers of L^θ are copies of T_{17} and T_{19}.*

(e) *If the top two layers of L are copies of T_{17} and T_{19}, then the top two layers of L^θ are copies of T_{14} and T_{20}, or T_{14} and T_{21}, according to whether the third layer of L has rank 7 or 8.*

(f) *If j successive layers of L form a tower of j copies of T_{18}, where $j \geq 2$, then the corresponding j layers of L^θ are either a tower of $j-1$ copies of T_{18} on top of a single copy of T_{19}, or a a tower of j copies of T_{18}, depending on whether or not the next layer of L contains a copy of T_{20}.*

(g) *If two successive layers of L are copies of T_{18} and T_{19}, then the corresponding layers of L^θ are two copies of T_{18}.*

(h) *If two successive layers of L are copies of T_{18} and T_{20}, then the corresponding layers of L^θ are two copies of T_{19}.*

(i) *If j is the largest non-negative integer for which j successive layers of L form a tower of copies of T_{19}, and $j \geq 2$, then the corresponding j layers of L^θ are a copy of T_{18}, followed by a tower of $j-2$ copies of T_{20}, and then a copy of T_{21}, unless the first layer of L is a copy of T_{17}, in which case the top layer of L^θ is a copy of T_{14}, and the next j layers of L^θ consist of a tower of $j-1$ copies of T_{20} followed by a copy of T_{21}.*

(j) *If j successive layers of L form a tower of j copies of T_{20}, where $j \geq 2$, then the corresponding j layers of L^θ are a tower of j copies of T_{19}.*

Proof. We will prove just some of this, and leave the rest for the reader. Most of it follows from observations about the θ-images of particular elements considered earlier. We can use those (and the θ-images of y_λ and $(y_{\lambda^2})^7$) to help us see what happens to layers of G_1-invariant subgroups of K under the action of θ.

First, suppose K/L has exponent $m = 7^\ell$, where $\ell \geq 2$. Then L^θ contains the elements v_i^7 and hence also the elements $v_i^{\frac{m}{7}}$, for $1 \leq i \leq 6$, since these lie in F. But also L contains w_j^m for $1 \leq j \leq 8$, and therefore L^θ must also contain $(w_j^m)^\theta = (w_j^\theta)^m$ for all such j. Now we know that $(w_8^7)^\theta = v_1^{-2}v_2^{-1}v_3^{-2}v_4^{-1}v_5^3v_6^3\, y_2^8\, (w_8^7)^{-3}$, and it follows that L^θ contains $(w_8^m)^\theta = ((w_8^7)^\theta)^{\frac{m}{7}} = (v_1^{-2}v_2^{-1}v_3^{-2}v_4^{-1}v_5^3v_6^3)^{\frac{m}{7}}\, y_2^{\frac{8m}{7}}\, (w_8^m)^{-3}$.

Hence the ℓth layer $(L^\theta)_{\ell-1}/(L^\theta)_\ell$ of L^θ contains the image of the subgroup generated by $V^{(\frac{m}{7})} \cup \{y_2^{\frac{8m}{7}}\}$, or equivalently, by $V^{(\frac{m}{7})} \cup \{y_2^{\frac{m}{7}}\}$. This is the same as the image of the subgroup generated by $V^{(\frac{m}{7})} \cup \{x_6^{\frac{m}{7}}\}$, by observations made a few paragraphs after Table 4.1, and so is a copy of T_{19}. Thus the ℓth layer of L^θ contains a copy of T_{19}, which proves part (a).

Now recall that we chose λ as a primitive root of 1 mod m, with $\lambda \equiv 2$ mod 7 (and $\lambda^2 \equiv 4$ mod 7). For m divisible by 49 this means $\lambda \equiv 30$ mod 49, while for m

divisible by 343 it means $\lambda \equiv 324 \bmod 343$, so that $\lambda = 2 + 7d$ for some integer d, with $d \equiv 4 \bmod 7$ when $\ell > 1$, and $d \equiv 46 \bmod 49$ when $\ell > 2$. Also $\lambda^2 = 4 + 7e$, where $e = 4d + 7d^2 \equiv 2 \bmod 7$ when $\ell > 1$, and $e \equiv 2 \bmod 49$ when $\ell > 2$.

By definition, we know that $y_\lambda = w_7 w_8^\lambda = w_7 w_8^{2+7d} = y_2 (w_8^7)^d$, and then similarly, we have $y_{\lambda^2} = w_7 w_8^{\lambda^2} = w_7 w_8^{4+7e} = y_2 w_8^{2+7e}$.

Using the θ-images of y_2 and w_8^7 we calculated earlier, we find that

$$\begin{aligned} y_\lambda^\theta &= (y_2 (w_8^7)^d)^\theta = y_2^\theta ((w_8^7)^\theta)^d \\ &= (v_2 v_3^{-1} v_4^{-1} v_5 \, y_2^3 w_8^{-7}) \, (v_1^{-2} v_2^{-1} v_3^{-2} v_4^{-1} v_5^3 v_6^3 \, y_2^8 (w_8^7)^{-3})^d \\ &= v_1^{-2d} v_2^{1-d} v_3^{-1-2d} v_4^{-1-d} v_5^{1+3d} v_6^{3d} \, y_2^{3+8d} w_8^{-7-21d}. \end{aligned}$$

Note that $3 + 8d \equiv 0 \bmod 7$ (and also $-7 - 21d \equiv 0 \bmod 7$), and so the image of y_λ^θ in $K/K^{(7)}$ lies in the image of the subgroup V (generated by v_1 to v_6).

The analogous property holds for higher powers of these elements, and so if some layer L_i/L_{i+1} of L is a copy of T_{19} (of rank 7), then the corresponding layer of L^θ can be a copy of T_{18} (of rank 6), depending on what happens with the layers above and below it.

On the other hand, $3 + 8d \equiv 28 \bmod 49$ while $-7 - 21d \equiv 7 \equiv 56 \bmod 49$, and so the image of $y_2^{3+8d} w_8^{-7-21d}$ in $K/K_2 = K/K^{(49)}$ is the same as the image of $(y_2 w_8^2)^{28}$, and then since $y_{\lambda^2} = w_7 w_8^{2+7e}$, this is the same as the image of $y_{\lambda^2}^{28}$. Hence if a layer of L is a copy of T_{19}, then the next layer of L^θ contains not only a copy of T_{18} but also the non-trivial image of a power of $x_8 = y_{\lambda^2}$, and therefore contains a copy of T_{20}, so must be a copy of T_{20} or T_{21}.

In fact we have more than that, because

$$\begin{aligned} x_6^\theta &= (v_6 y_\lambda^2)^\theta = v_6^\theta (y_\lambda^\theta)^2 \\ &= (v_1 v_4^{-1} v_5^{-1}) \, (v_1^{-2d} v_2^{1-d} v_3^{-1-2d} v_4^{-1-d} v_5^{1+3d} v_6^{3d} \, y_2^{3+8d} w_8^{-7-21d})^2 \\ &= v_1^{1-4d} v_2^{2-2d} v_3^{-2-4d} v_4^{-3-2d} v_5^{1+6d} v_6^{6d} \, y_2^{6+16d} w_8^{-14-42d} \\ &= (v_1 v_4)^{1-4d} (v_2 v_4^{-1})^{2-2d} (v_3 v_4^{-2-4d})^{-3-2d} (v_5 v_4^{-2})^{1+6d} (v_6 v_4^3)^{6d} \, v_4^{-2-10d} \\ &\qquad y_2^{6+16d} w_8^{-14-42d}. \end{aligned}$$

Noting that $-2 - 10d$, $6 + 16d$ and $-14 - 42d$ are all divisible by 7, we see from this that the image of x_6^θ in $K/K^{(7)}$ lies in the image of the subgroup generated by $v_1 v_4$, $v_2 v_4^{-1}$, $v_3 v_4^{-1}$, $v_5 v_4^{-2}$ and $v_6 v_4^3$, namely T_{14}.

Hence if the top layer of L is a copy of T_{17} (which is generated by T_{14} and the image of x_6), then the top layer of L^θ can be a copy of T_{14}. On the other hand, the second layer of L^θ contains a copy of T_{18} and the image of of $(y_2 w_8^2)^{28}$, and hence contains a copy of T_{20}, so must be a copy of T_{20} or T_{21}.

It follows, for example, that if the top two layers of L are copies of T_{17} and T_{19}, then the top layer of L^θ contains a copy of T_{14} and the second layer contains a copy of T_{20}. In fact, since we are assuming that K/L has exponent $m = 7^\ell$, and T_{19} has rank 7, all of the next $\ell - 1$ layers of L after the first one will be copies of T_{19}, and so all of the corresponding layers of L^θ must contain copies of T_{20}. Also by (a), the ℓth layer of L^θ contains a copy of T_{19} as well, and hence must have rank 8. In particular, $|K:L| = |T_{21}:T_{17}||T_{21}:T_{19}|^{\ell-1} = 7^{\ell+1}$, while $|K:L^\theta| \le |T_{21}:T_{14}||T_{21}:T_{20}|^{\ell-2} = 7^{\ell+1}$, and since we know that $|K:L| = |K:L^\theta|$, this forces the top layer of L^θ to be T_{14} and all of the next $\ell-2$ layers to be T_{20}. In particular, this proves part (e). The proof of part (i) is similar.

Next, $x_8 = y_{\lambda^2} = w_7 w_8^{\lambda^2}$ $(= w_7 w_8^{4+7e} = y_2 w_8^{2+7e})$, and therefore

$$\begin{aligned}
x_8^\theta &= ((y_{\lambda^2})^7)^\theta = ((w_7 w_8^{\lambda^2})^7)^\theta = (w_7^7)^\theta\,((w_8^7)^\theta)^{\lambda^2} \\
&= (v_1^{-3}v_2^2v_3^{-3}v_4^2v_5v_6\,y_2^5\,(w_8^7)^{-1})\,(v_1^{-2}v_2^{-1}v_3^{-2}v_4^{-1}v_5^3v_6^3\,y_2^8\,(w_8^7)^{-3})^{\lambda^2} \\
&= v_1^{-3-2\lambda^2}v_2^{2-\lambda^2}v_3^{-3-2\lambda^2}v_4^{2-\lambda^2}v_5^{1+3\lambda^2}v_6^{1+3\lambda^2}\,y_2^{5+8\lambda^2}(w_8^7)^{-1-3\lambda^2} \\
&= (v_1v_4)^{-3-2\lambda^2}(v_2v_4^{-1})^{2-\lambda^2}(v_3v_4^{-1})^{-3-2\lambda^2}(v_5v_4^{-2})^{1+3\lambda^2}(v_6v_4^3)^{1+3\lambda^2}\,v_4^{3-5\lambda^2} \\
&\qquad y_2^{5+8\lambda^2}(w_8^7)^{-1-3\lambda^2}.
\end{aligned}$$

In this case $3-5\lambda^2 \equiv -7 \equiv 0 \bmod 7$ while $5+8\lambda^2 \equiv 37 \not\equiv 0 \bmod 7$, and so the image of x_8^θ in $K/K^{(7)}$ lies in the subgroup generated by the images of v_1v_4, $v_2v_4^{-1}$, $v_3v_4^{-1}$, $v_5v_4^{-2}$, $v_6v_4^3$ and y_2, which is T_{17}.

Hence if some layer of L is copy of T_{20} (generated by the images of V and x_8), then the next layer up in L^θ contains a copy of T_{17} and so must be T_{17} or T_{19}. This cannot be a copy of T_{21}, by (a), and moreover, it is a copy of T_{17} only if those layers are the second layer of L and the top layer of L^θ. In all other cases it is a copy of T_{19}.

Proofs of parts (d), (h) and (j) follow easily from this, and proofs of the remaining parts are similar to these and the ones completed above. $\square$

The observations in the above propoosition now make it easy to determine all of the G_2^1-invariant subgroups of finite prime-power index in K.

For example, if L has type $100_{(1,49)}$, with the first two layers being copies of T_{14} and T_{19} and all subsequent layers having rank 8, then it follows from part (c) that L^θ has the same type, and hence L is preserved by θ. On the other hand, if L has type $100_{(1,343)}$, with the first three layers being copies of T_{14}, T_{19} and T_{19}, and all subsequent layers having rank 8, then it follows from part (i) that the first three layers of L^θ are copies of T_{14}, T_{18} and T_{21}, and so L is not preserved by θ.

Thus we obtain the following, which will also be used shortly when we consider isomorphisms between the covers:

Corollary 6.2 *The effect of θ on the G_1-invariant subgroups of K lying between F and J is as described in Table* 6.2.

Type of L	Type of L^θ
45_7	$106_{(1,49)}$
$58_{(1,49)}$	$100_{(1,343)}$
$58_{(1,7^w)}$, where $w \geq 3$	$104_{(1,7^{w-1},7^{w+1})}$
59_7	59_7
$60_{(1,7)}$	$109_{(1,49)}$
$60_{(1,7^w)}$, where $w \geq 2$	$107_{(1,7^{w-1},7^{w+1})}$
$100_{(1,49)}$	$100_{(1,49)}$
$100_{(1,343)}$	$58_{(1,49)}$
$100_{(1,7^w)}$, where $w \geq 4$	$105_{(1,49,7^{w-1})}$
$103_{(1,7^w)}$, where $w \geq 2$	$106_{(1,7^{w+1})}$
$104_{(1,7^{w-1},7^w)}$, where $w \geq 3$	$104_{(1,7^{w-1},7^w)}$
$104_{(1,7^{w-2},7^w)}$, where $w \geq 4$	$58_{(1,7^{w-1})}$
$104_{(1,7^v,7^w)}$, where $w-3 \geq v \geq 2$	$105_{(1,7^{v+1},7^{w-1})}$
$105_{(1,49,7^w)}$, where $w \geq 3$	$100_{(1,7^{w+1})}$
$105_{(1,7^v,7^w)}$, where $w > v > 2$	$104_{(1,7^{v-1},7^{w+1})}$
$106_{(1,49)}$	45_7
$106_{(1,7^w)}$, where $w \geq 3$	$103_{(1,7^{w-1})}$
$107_{(1,7^{w-1},7^w)}$, where $w \geq 2$	$107_{(1,7^{w-1},7^w)}$
$107_{(1,7^{w-2},7^w)}$, where $w \geq 3$	$60_{(1,7^{w-1})}$
$107_{(1,7^v,7^w)}$, where $w-3 \geq v \geq 1$	$108_{(1,7^{v+1},7^{w-1})}$
$108_{(1,7,7^w)}$, where $w \geq 2$	$109_{(1,7^{w+1})}$
$108_{(1,7^v,7^w)}$, where $w > v > 1$	$107_{(1,7^{v-1},7^{w+1})}$
$109_{(1,7)}$	$109_{(1,7)}$
$109_{(1,49)}$	$60_{(1,7)}$
$109_{(1,7^w)}$, where $w \geq 3$	$108_{(1,7,7^{w-1})}$

Table 6.2: Effect of θ on the G_1-invariant subgroups from Table 6.1

In particular, this gives us all of the G_1-invariant subgroups of prime-power index in K that are also G_2^1-invariant:

Corollary 6.3 *The G_2^1-invariant subgroups of finite prime-power index in K are the following, from Table* 6.1:

- *the subgroup of type* $109_{(1,7)}$*, which is J, with quotient $K/L \cong \mathbb{Z}_7$,*
- *the subgroup of type* 59_7*, generated by $F \cup \{z_{\lambda^2}\}$, with quotient $K/L \cong (\mathbb{Z}_7)^2$,*
- *the subgroup of type* $100_{(1,49)}$*, with quotient $K/L \cong (\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{49}$,*
- *the subgroup of type* $107_{(1,7^{\ell-1},7^\ell)}$*, with quotient $K/L \cong \mathbb{Z}_{7^{\ell-1}} \oplus \mathbb{Z}_{7^\ell}$, for each $\ell \geq 2$,*
- *one of the subgroups of type* ${104_{(1,7^{\ell-1},7^\ell)}}^*$*, with quotient $K/L \cong \mathbb{Z}_7 \oplus \mathbb{Z}_{7^{\ell-1}} \oplus \mathbb{Z}_{7^\ell}$, for each $\ell \geq 3$.*

Note that none of these subgroups has top layer isomorphic to T_{12} or T_{21}, and so none of them can be G_4^1-invariant, but actually that follows also from the fact that no finite symmetric cubic graph admits both a 2-arc-regular and a 4-arc-regular group of automorphisms (see [6, Theorem 3]).

We still need to check for G_3-invariance, but this is easy:

By [6, Proposition 26] or [5, Proposition 2.3], if the regular cover resulting from a G_1-invariant subgroup L has a 3-arc-regular group of automorphisms, then it must also admit a 2-arc-regular group of automorphisms, and so L must come from the restricted set of G_2^1-invariant possibilities that we found above. On the other hand, the group G_3 can be obtained from G_2^1 by adjoining the involutory automorphism τ that interchanges h, a and θ with h, $a\theta$ and θ (respectively). This automorphism τ interchanges $(ha)^2$ with $hah^{-1}a$, and $(h^{-1}a)^2$ with $h^{-1}aha$, and hence takes the element $v_1 = w_1 = (ha)^6$ to $(hah^{-1}a)^3 = w_5 w_1^{-1} hahah$, which does not lie in K, let alone in any subgroup L of K. Similarly, τ takes $v_1 v_4 = w_1 w_4 w_8^{-1}$ to $w_5 w_1^{-1} w_7^{-1} = v_1^{-1} v_5$, but the image of this in $K/K^{(7)}$ does not lie in the subgroup T_{14}, so τ does not preserve any G_2^1-invariant subgroup L with T_{14} as its top layer. Hence τ preserves no G_2^1-invariant subgroup of finite index, and therefore we have no 3-arc-regular cover.

Finally, we determine isomorphisms between the covering graphs that arise from the G_1-invariant subgroups we have found.

When the subgroup L is G_4^1-invariant, the cover is 4-arc-regular, and unique up to isomorphism, since the subgroup K is normal in G_4^1 but not in G_5. Similarly, when the subgroup is G_2^1-invariant, the cover is 2-arc-regular, and unique up to isomorphism, since K is normal in G_1 but not in G_2^1.

So now suppose L is G_1-invariant, but not G_2^1- or G_4^1-invariant. Then the cover obtained from L will be unique up to isomorphism unless there exists an outer automorphism of G_1 taking L to another G_1-invariant subgroup of K. Let us suppose that happens.

The group G_1 is the modular group $\mathrm{PSL}(2,\mathbb{Z})$, and isomorphic to the free product $C_2 \star C_3$, so (as is well known) the automorphism group of G_1 is the group G_2^1, generated by G_1 and the involutory automorphism θ that inverts the two standard generators of G_1, and in particular, $G_2^1 \cong \mathrm{PGL}(2,\mathbb{Z})$. Hence we may suppose the outer automorphism takes L to L^θ. In particular, since L^θ lies in K, we find that L must be one of the subgroups described in Table 6.1, but not one of those that are preserved by θ.

It follows that if L is a G_1-invariant subgroup of J containing F (in which case L will certainly not be G_4^1-invariant), then either $L = L^\theta$ and the cover is 2-arc-regular, or $L^\theta \neq L$ but L and L^θ define the same 1-arc-regular cover of the Heawood graph. Note, however, that in the latter case, the exponents of K/L and K/L^θ are always different — in fact one of them is always 7 times the other — so we do not have to take much account of them when enumerating all possibilities for L such that the covering group K/L has given exponent.

In all other cases, where L does not contain F or is not contained in J, the cover is unique up to isomorphism.

7 Main theorem

Thus we have the following, with 'for each $d \,|\, m$' and 'for each $d \,||\, m$' meaning 'for each divisor d of m' and 'for each proper divisor d of m', respectively:

Theorem 7.1 *Let $m = k^\ell$ be any power of a prime k, with $\ell > 0$. Then the arc-transitive abelian regular covers of the Heawood graph with covering group of exponent m are as follows:*

(a) *If $k \equiv 2 \bmod 3$, there are exactly $2\ell + 1$ such covers, namely*
- *one 4-arc-regular cover with covering group $(\mathbb{Z}_m)^8$,*
- *one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_m)^6$ and one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^6 \oplus (\mathbb{Z}_m)^2$, for each $d \,||\, m$.*

(b) *If $k \equiv 1 \bmod 3$ and $k \neq 7$, there are exactly $3\ell^2 + 3\ell + 1$ such covers, namely*
- *one 4-arc-regular cover with covering group $(\mathbb{Z}_m)^8$,*
- *two 1-arc-regular covers with covering group $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_m)^6$ and one 1-arc-regular cover with covering group $(\mathbb{Z}_c)^6 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_m$, for each ordered pair (c, d) of distinct divisors of m.*

(c) *If $k = 3$, there are exactly $4\ell + 1$ such covers, namely*
- *two 4-arc-regular covers, with covering groups $(\mathbb{Z}_m)^8$ and $\mathbb{Z}_{\frac{m}{3}} \oplus (\mathbb{Z}_m)^7$,*
- *one 1-arc-regular cover with covering group $\mathbb{Z}_d \oplus \mathbb{Z}_{3d} \oplus (\mathbb{Z}_m)^6$ for each $d \,||\, \frac{m}{3}$,*
- *one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_m)^6$, one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^6 \oplus (\mathbb{Z}_m)^2$, and one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^6 \oplus \mathbb{Z}_{\frac{m}{3}} \oplus \mathbb{Z}_m$, for each $d \,||\, m$.*

(d) *If $k = 7$ and $\ell \geq 3$, there are exactly $54\ell^2 - 54\ell + 14$ such covers, namely*

- *two* 4*-arc-regular covers, with covering groups* $(\mathbb{Z}_m)^8$ *and* $(\mathbb{Z}_{\frac{m}{7}})^5 \oplus (\mathbb{Z}_m)^3$,
- *two* 2*-arc-regular covers, with covering groups* $\mathbb{Z}_7 \oplus \mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_m$ *and* $\mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_m$,
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_m)^7$*, for each* $d \,||\, m$,
- *three* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus \mathbb{Z}_{\frac{m}{7}} \oplus (\mathbb{Z}_m)^6$*, for each* $d \,||\, \frac{m}{7}$,
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_m)^6$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^2 \oplus (\mathbb{Z}_m)^6$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_m)^6$*, for each* $d \,||\, \frac{m}{7}$,
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{m}{7}} \oplus (\mathbb{Z}_m)^5$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *three* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^2 \oplus (\mathbb{Z}_m)^5$*, for each* $d \,||\, \frac{m}{7}$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus \mathbb{Z}_{\frac{m}{7}} \oplus (\mathbb{Z}_m)^5$*, for each* $d \,||\, \frac{m}{7}$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^3 \oplus (\mathbb{Z}_m)^5$,
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^2 \oplus (\mathbb{Z}_m)^4$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *three* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^3 \oplus (\mathbb{Z}_m)^4$*, for each* $d \,||\, \frac{m}{7}$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^2 \oplus (\mathbb{Z}_m)^4$*, for each* $d \,||\, \frac{m}{7}$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^4$,
- *fourteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^3 \oplus (\mathbb{Z}_m)^3$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *fifteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^3$*, for each* $d \,||\, \frac{m}{7}$,
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^3 \oplus (\mathbb{Z}_m)^3$*, for each* $d \,||\, \frac{m}{7}$,
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^5 \oplus (\mathbb{Z}_m)^3$,
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^2$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *nine* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^5 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$,
- *seven* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{49}$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{49}})^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^2$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{49}$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{49}$,

• *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$*,*

• *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$*,*

• *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{7d} \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$*, but with one of these for* $d = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*

• *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^6 \oplus (\mathbb{Z}_m)^2$*,*

• *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^6 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$*, but with the one for* $d = 1$ *having* $\mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*

• *fifteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus \mathbb{Z}_{\frac{m}{49}} \oplus (\mathbb{Z}_{\frac{m}{7}})^5 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{49}$*,*

• *fourteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^5 \oplus \mathbb{Z}_m$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{343}$*,*

• *eight* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_m$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{343}$*,*

• *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*

• *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*

• *fourteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*

• *fifteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_{49d} \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{49}$*,*

• *fourteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_{7c})^5 \oplus \mathbb{Z}_{49d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{343}$ *with* $c < d$*,*

• *eight* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_{49d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{343}$ *with* $c < d$*,*

• *nine* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^6 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*,*

• *nine* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^6 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{49}$*,*

• *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_d)^6 \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{343}$ *with* $c < d$*,*

• *nine* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*,*

• *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^5 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{49}$*,*

• *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_{7c})^4 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*

• *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*,*

- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_{7c})^3 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^4 \oplus (\mathbb{Z}_{7c})^2 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*
- *fifteen* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*, but with one of those for* $d = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49} \oplus \mathbb{Z}_{\frac{m}{49}}$ *as an alternative covering group, and another of those for* $d = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*
- *thirteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus \mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_m$*,*
- *fourteen* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^5 \oplus \mathbb{Z}_{7c} \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$ *other than* $(1, \frac{m}{49})$*, but with one of those for each pair* (c, d) *with* $c = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49d} \oplus \mathbb{Z}_{\frac{m}{7}}$ *as an alternative covering group, and another one of those for each pair* (c, d) *with* $c = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^6 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*, but with the three such covers in the case* $d = 1$ *having (in some order) respectively* $(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{\frac{m}{7}}$*,* $\mathbb{Z}_{49} \oplus \mathbb{Z}_{\frac{m}{7}}$ *and* $\mathbb{Z}_{7m}$ *as an alternative covering group,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^6 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$ *other than* $(1, \frac{m}{49})$*, but with the two such covers in each case with* $c = 1$ *having (in some order) respectively* $\mathbb{Z}_{49d} \oplus \mathbb{Z}_{\frac{m}{7}}$ *and* $\mathbb{Z}_d \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*
- *one* 1*-arc-regular cover with covering group* $\mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_m$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^7 \oplus \mathbb{Z}_m$*, for each* $d \,||\, m$*, but with one of those for* $d = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{\frac{m}{7}}$ *as an alternative covering group.*

(e) *If* $k = 7$ *and* $e = 2$ *(so that* $m = 49$*), there are exactly* 122 *such covers, namely*

- *two* 4*-arc-regular covers, with covering groups* $(\mathbb{Z}_{49})^8$ *and* $(\mathbb{Z}_7)^5 \oplus (\mathbb{Z}_{49})^3$*,*
- *two* 2*-arc-regular covers, with covering groups* $(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{49}$ *and* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49}$*,*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus (\mathbb{Z}_{49})^7$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{49})^7$*,*
- *three* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus (\mathbb{Z}_{49})^6$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^2 \oplus (\mathbb{Z}_{49})^6$*,*
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_{49})^6$*,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^2 \oplus (\mathbb{Z}_{49})^5$*,*
- *one* 1*-arc-regular cover with covering group* $\mathbb{Z}_7 \oplus (\mathbb{Z}_{49})^5$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3 \oplus (\mathbb{Z}_{49})^5$*,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3 \oplus (\mathbb{Z}_{49})^4$*,*
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_7)^2 \oplus (\mathbb{Z}_{49})^4$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4 \oplus (\mathbb{Z}_{49})^4$*,*

- *fifteen* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4 \oplus (\mathbb{Z}_{49})^3$,
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3 \oplus (\mathbb{Z}_{49})^3$,
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^5 \oplus (\mathbb{Z}_{49})^3$,
- *nine* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^5 \oplus (\mathbb{Z}_{49})^2$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4 \oplus (\mathbb{Z}_{49})^2$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_7)^3 \oplus (\mathbb{Z}_{49})^2$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_7)^2 \oplus (\mathbb{Z}_{49})^2$,
- *seven* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus (\mathbb{Z}_{49})^2$, *but with one of these having* $(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{343}$ *as an alternative covering group,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^6 \oplus (\mathbb{Z}_{49})^2$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_{49})^2$, *but with one of these having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{343}$ *as an alternative covering group,*
- *nine* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^6 \oplus \mathbb{Z}_{49}$,
- *nine* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^5 \oplus \mathbb{Z}_{49}$,
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4 \oplus \mathbb{Z}_{49}$,
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3 \oplus \mathbb{Z}_{49}$,
- *fourteen* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{49}$, *but with one of these having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{343}$ *as an alternative covering group,*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49}$, *but with one of these having* $(\mathbb{Z}_7)^3$ *as an alternative covering group, and the other having* $\mathbb{Z}_{343}$ *as an alternative covering group,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^7 \oplus \mathbb{Z}_{49}$, *and*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_{49}$, *but with one of these having* $(\mathbb{Z}_7)^2$ *as an alternative covering group.*

(f) *If* $k = 7$ *and* $e = 1$ (*so that* $m = 7$), *there are exactly* 21 *such covers, namely*

- *two* 4*-arc-regular covers, with covering groups* $(\mathbb{Z}_7)^8$ *and* $(\mathbb{Z}_7)^3$,
- *two* 2*-arc-regular covers, with covering groups* $\mathbb{Z}_7$ *and* $(\mathbb{Z}_7)^2$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^7$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^6$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^5$,
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4$,
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3$, *but with one of these having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49}$ *as an alternative covering group,*
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_7)^2$, *but also having* $\mathbb{Z}_{49}$ *as an alternative covering group, and*
- *one* 1*-arc-regular cover with covering group* $\mathbb{Z}_7$.

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