Arc-transitive abelian regular covers of cubic graphs

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Abstract

Quite a lot of attention has been paid recently to the construction of edge- or arc-transitive covers of symmetric graphs. In most cases, the approach has involved voltage graph techniques, which are excellent for finding regular covers in which the group of covering transformations is either cyclic or elementary abelian, or more generally, homocyclic, but are not so easy to use when the covering group has other forms — even when it is abelian but not homocyclic. In this paper, a different approach is introduced that can be used more widely. This new approach takes a universal group for the action of the automorphism group of the base graph, and uses Reidemeister-Schreier theory to obtain a presentation for a 'universal covering group', and some representation theory and other methods for determining suitable quotients. This approach is then used to find all arc-transitive abelian regular covers of K_4, $K_{3,3}$, the cube Q_3, and the Petersen graph. A sequel will do the same for the Heawood graph.

1 Introduction

Various constructions for regular coverings of graphs are well known and helpful in both algebraic and topological graph theory. In particular, constructions using the cycle space of the graph and voltage graph techniques have been used to construct or classify graphs with particular symmetry properties.

For example, independently in the 1970s, Conway (see [1]) and Djoković [11] used graph covers to prove the existence of infinitely many 5-arc-transitive cubic graphs, as covers of Tutte's 8-cage. Also Djoković [11] used lifts of automorphisms along covering projections to show that if an s-arc-transitive group of automorphisms of a graph can be lifted along a regular covering projection, then the covering graph is at least s-arc-transitive. Subsequently Biggs [2] developed a method for constructing certain 5-arc-transitive cubic graphs as covers of cubic graphs that are 4- but not 5-arc-transitive.

The theory and use of voltage graphs was developed by Gross and Tucker [19], and used for example to show that every covering graph can be realised as a voltage graph construction.

Some years later, Malnič, Marušič and Potočnik [27] took these ideas further in a systematic study of regular covering projections of a given connected graph along which a given group of automorphisms lifts, and used this to give an explicit means of construction of such coverings when the covering group is elementary abelian. Their approach involves taking an appropriate representation of automorphisms of the base graph (by matrices), and then converting the conditions for lifting into a problem of finding invariant subspaces of certain concrete groups of matrices over prime fields.

The approach developed in [27] has been successfully applied to the classification of elementary abelian covers of a number of symmetric graphs of small valency, including the Petersen graph [29], the Heawood graph [27], the Möbius-Kantor graph [26], the Pappus graph [30], the octahedron graph [22], and the complete graph K_5 [21].

A similar approach, using linear criteria for lifting automorphisms, was developed by Du, Kwak and Xu [13] in order to find symmetric regular covers of small cubic graphs with cyclic or elementary abelian covering group, and this has been applied to find such covers for the complete graph K_4 [16], the 3-dimensional cube graph Q_3 [17], the complete bipartite graph $K_{3,3}$ [14], and the Petersen graph [15].

The above methods are excellent for finding symmetric regular covers in which the group of covering transformations is either cyclic or elementary abelian, or more generally, homocyclic (that is, isomorphic to a direct sum $(\mathbb{Z}_m)^r = \mathbb{Z}_m \oplus Z_m \oplus \cdots \oplus Z_m$ of rank $r > 1$ and composite exponent m). But they are not so easy to use when the covering group has other forms — even when it is abelian but not homocyclic.

In this paper, a different approach is introduced that can be used more widely. This new approach takes a universal group for the action of the automorphism group of the base graph, uses Reidemeister-Schreier theory to obtain a presentation for a 'universal covering group', and then uses some representation theory and other methods for determining suitable quotients. We illustrate this approach by its application to find all arc-transitive abelian regular covers of K_4, $K_{3,3}$, the cube Q_3, and the Petersen graph.

We begin with some further background in Section 2, and then describe the new approach in Section 3. Then we deal with K_4, $K_{3,3}$, Q_3 and the Petersen graph in Sections 4 to 7. The Heawood graph will be covered in a sequel.

2 Preliminaries

In this Section we give some further background, including some details that we require later. Much of the material on graph covers and symmetries can be found in [9] or [27], and much of that on symmetric cubic graphs in [8], [10] or [12]. Throughout the paper, every graph X will be finite, undirected, simple and connected, and $V(X)$, $E(X)$ and $A(X)$ will denote the vertex-set, edge-set and arc-set of X, respectively.

2.1 Graph covers and symmetries

A *graph homomorphism* from a graph Y to a graph X is a mapping $p\colon V(Y) \to V(X)$ preserving adjacency, that is, with $\{p(u), p(v)\} \in E(X)$ for all $\{u, v\} \in E(Y)$. If such a mapping p is surjective, then X is called a *quotient* (or homomorphic image) of Y, and if p is surjective and also locally bijective (preserving valence at each vertex), then Y is called a *cover* of X, and p is called a *covering projection*. In the latter case, we sometimes call X the *base graph*, and the pre-image $p^{-1}(v) = \{\, u \in V(Y) \mid p(u) = v \,\}$ the *fibre* of p over v, for each $v \in V(X)$.

An *automorphism* of a graph X is a bijective graph homomorphism from X to X. Under composition, the automorphisms of X form a group, called the *automorphism group* of X and denoted by $\mathrm{Aut}\,X$. The connected graph X is called *symmetric* (or *arc-transitive*) if $\mathrm{Aut}\,X$ is transitive on $A(X)$, in which case $\mathrm{Aut}\,X$ is also transitive on both $V(X)$ and $E(X)$, and therefore X has to be regular (in the sense of all vertices having the same valence). A subgroup L of $\mathrm{Aut}\,X$ is called *semi-regular* if the stabilizer in L of any vertex or arc of X is trivial — that is, if L acts regularly on each of its orbits on $V(X)$ and $A(X)$.

Next, we consider the notion of *lifting*. Let $p\colon Y \to X$ be a covering projection, and suppose α and β are automorphisms of X and Y such that $\alpha \circ p = p \circ \beta$, that is, such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\;p\;} & X \\ \beta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow[\;p\;]{} & X \end{array}$$

Then we say α *lifts* to β, and β *projects* to α, and also we call β a *lift* of α, and α a *projection* of β. Note that α is uniquely determined by β, but β is not generally determined by α. The set of all lifts of a given $\alpha \in \mathrm{Aut}\,X$ is denoted by $L(\alpha)$. If every automorphism of a subgroup G of $\mathrm{Aut}\,X$ lifts (to an automorphism of Y), then $\bigcup_{\alpha \in G} L(\alpha)$ is a subgroup of $\mathrm{Aut}\,Y$, called the *lift* of G. In particular, the lift of the identity subgroup of $\mathrm{Aut}\,X$ (or equivalently, the subgroup of all automorphisms of Y that project to the identity automorphism of X) is called the *group of covering transformations*, or *voltage group*, and sometimes denoted by $\mathrm{CT}(p)$.

For any semi-regular subgroup N of $\mathrm{Aut}\, X$, we may define a quotient graph X_N by taking vertices as the orbits of N on $V(X)$ and edges as the orbits of N on $E(X)$, with the obvious incidence. When we do this, the orbit-representative mapping from $V(X)$ to $V(X_N)$ becomes a covering projection, with N as its group of covering transformations. This motivates the definition of a *regular covering projection*:

A covering projection $p\colon Y \to X$ is called *regular* if there exists a semi-regular subgroup N of $\mathrm{Aut}\, Y$ such that the quotient graph Y_N is isomorphic to X. In that case, we call Y a *regular cover* of X, with covering transformation group N. Also the regular cover Y and the regular covering projection p are called (for example) abelian, or cyclic, or elementary abelian, or homocyclic, if the group N is abelian, or cyclic, or elementary abelian, or homocyclic, respectively.

The normalizer of N in $\mathrm{Aut}\, Y$ projects to the largest subgroup of $\mathrm{Aut}\, X$ that lifts. Hence in particular, if the latter subgroup is B, say, then the lift of B has a normal subgroup N (the covering transformation group) with quotient isomorphic to B.

In this paper, we are interested in the situation where the cover Y is symmetric (in which case the base graph X must be symmetric as well). Our approach will involve taking an arc-transitive subgroup B of $\mathrm{Aut}\, X$, and finding all possible ways of lifting B to an arc-transitive group of automorphisms of a regular cover Y of X, with abelian covering transformation group. In simple terms, we find all symmetric regular covers of the given base graph X.

Usually, regular covers are studied up to equivalence. Two regular covering projections $p : Y \to X$ and $p' : Y' \to X$ are called *equivalent* if there exists a graph isomorphism $\theta\colon Y \to Y'$ such that $p = p'\theta$. In this paper, however, we will simply classify the possibilities for Y (for a given X), up to isomorphism, as the projection p is uniquely determined in the cases we consider.

2.2 Symmetric cubic graphs

An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that v_i is adjacent with v_{i+1} for $0 \le i < s$, and $v_{i-1} \ne v_{i+1}$ for $0 < i < s$, or in other words, such that any two consecutive vertices are adjacent, and any three consecutive vertices are distinct. A graph X is *s-arc-transitive* if its automorphism group acts transitively on the set of s-arcs of X, and *s-arc-regular* if this action is sharply-transitive. In particular, 0-arc-transitive means vertex-transitive, while 1-arc-transitive means arc-transitive (or symmetric). Note that under the connectedness assumption, s-arc-transitivity implies $(s-1)$-arc-transitivity, for all $s \ge 1$.

The simple cycle graph C_n is s-arc-transitive for all $s \ge 0$. The complete graph K_n is vertex-, edge- and arc-transitive for all $n \ge 3$, but is 2-arc-transitive only when $n = 3$ (as there are two types of 2-arc when $n \ge 4$), and similarly, the complete bipartite graph $K_{n,n}$ is 3-arc- but not 4-arc-transitive, for all $n \ge 2$.

A lot is known about symmetric graphs that are 3-valent, or *cubic*, thanks mostly to two seminal theorems of Tutte [31, 32], showing that there are no finite 6-arc-transitive cubic graphs, and that every finite symmetric cubic graph is s-arc-regular for some $s \leq 5$. For example, K_4 and Q_3 are 2-arc-regular, $K_{3,3}$ and the Petersen graph are 3-arc-regular, the Heawood graph (the incidence graph of the Fano plane) is 4-arc-regular, and Tutte's 8-cage (on 30 vertices) is 5-arc-regular.

Tutte's work was taken further by Goldschmidt, Sims, Djoković and others, and we now know that the automorphism group of every finite symmetric cubic graph is a quotient of one of seven finitely-presented groups, which can be listed as G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5, and presented as follows (see [12, 8]):

$G_1 = \langle\, h, a \mid h^3 = a^2 = 1 \,\rangle$ (the modular group);

$G_2^1 = \langle\, h, p, a \mid h^3 = p^2 = a^2 = 1,\ php = h^{-1},\ a^{-1}pa = p \,\rangle$;

$G_2^2 = \langle\, h, p, a \mid h^3 = p^2 = 1,\ a^2 = p,\ php = h^{-1},\ a^{-1}pa = p \,\rangle$;

$G_3 = \langle\, h, p, q, a \mid h^3 = p^2 = q^2 = a^2 = 1,\ pq = qp,\ php = h,\ qhq = h^{-1},\ a^{-1}pa = q \,\rangle$;

$$G_4^1 = \langle\, h, p, q, r, a \mid h^3 = p^2 = q^2 = r^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ (qr)^2 = p,\\ h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle;$$

$$G_4^2 = \langle\, h, p, q, r, a \mid h^3 = p^2 = q^2 = r^2 = 1,\ a^2 = p,\ pq = qp,\ pr = rp,\ (qr)^2 = p,\\ h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle;$$

$$G_5 = \langle\, h, p, q, r, s, a \mid h^3 = p^2 = q^2 = r^2 = s^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ ps = sp,\\ qr = rq,\ qs = sq,\ (rs)^2 = pq,\ h^{-1}ph = p,\ h^{-1}qh = r,\\ h^{-1}rh = pqr,\ shs = h^{-1},\ a^{-1}pa = q,\ a^{-1}ra = s \,\rangle.$$

Here the group G_s or G_s^i is a universal group for the situation where some group G of automorphisms of the graph acts regularly on s-arcs, with $i = 1$ or 2 depending on whether or not the group contains an involution a that reverses an arc (and $i = 1$ when s is odd). In that case, G is a smooth quotient of G_s or G_s^i, where 'smooth' here means that the orders of the generators are preserved.

Conversely, every smooth epimorphism from G_s or G_s^i to a finite group G gives rise to a connected cubic graph on which G acts as a group of automorphisms, with a single orbit on s-arcs. The vertices of this graph may be identified with cosets of a certain subgroup H, namely the image of $\langle h\rangle$ in the case of G_1, or $\langle h, p\rangle$ in the case of G_2^1 and G_2^2, or $\langle h, p, q\rangle$ in the case of G_3, or $\langle h, p, q, r\rangle$ in the case of G_4^1 and G_4^2, or $\langle h, p, q, r, s\rangle$ in the case of G_5, and the vertices Hx and Hy are adjacent if and only if xy^{-1} lies in the double coset HaH. The group G acts by right multiplication on cosets, and the subgroup H is then the stabilizer in G of a vertex. The graph itself may be called the *double coset graph* $X(G, H, a)$.

These observations were exploited in [8] to produce the first known examples of finite symmetric cubic graphs of the types corresponding to G_2^2 and G_4^2 (having no involutory automorphism flipping an edge), and in [7] to determine all symmetric cubic graphs of order up to 768. In what follows, we will say that a finite symmetric cubic graph X belongs to the class s or class s^i (for $i = 1$ or 2) if $\mathrm{Aut}\, X$ is a smooth quotient of G_s or G_s^i respectively.

Possible inclusions among arc-transitive subgroups of the automorphism group of a finite symmetric cubic graph were considered in [12] and taken further in [10]. For example, it is known that each of the groups G_2^1, G_3, G_4^1 and G_5 contains a subgroup isomorphic to G_1 (with index 2, 4, 8 and 16 respectively), corresponding to the fact that some 2-, 3-, 4- or 5-arc-regular cubic graphs admit a group of automorphisms acting regularly on the 1-arcs. On the other hand, the group G_5 contains no subgroup isomorphic to G_2^1 or G_2^2, so no finite 5-arc-regular cubic graph admits a group of automorphisms acting regularly on the 2-arcs. We will use such theorems (from [10, 12]) in later sections, without quoting them all here.

3 A new approach

Most of the approaches taken up to now for finding symmetric regular covers of symmetric graphs with abelian covering group have used voltage graphs or other methods to determine all possibilities for the cover with a given covering group, such as the cyclic group $\mathbb{Z}_p$ or the elementary abelian p-group $(\mathbb{Z}_p)^s$ of rank s. The approach we introduce here differs from these, in that it does not require a choice of the covering group in advance.

Furthermore, our approach is quite general, in that it can also be taken when considering covers of other structures with large automorphism groups, such as regular maps, Hurwitz surfaces (and other compact Riemann surfaces or with large automorphism group), and abstract regular polytopes, or indeed whenever there exist universal groups for the kinds of group action of interest.

Our approach was described briefly by the first author at the AGTAGC 2010 workshop (on algebraic, topological and complexity aspects of graph covers) in Auckland in February 2010. It was largely motivated by work by Leech [23] in determining the structure of certain normal subgroups of the $(2, 3, 7)$ triangle group Δ and the actions by conjugation of the generators of Δ on generators of those subgroups, and the use of these by Cohen [4] to classify abelian covers of Klein's quartic surface (or equivalently, all regular maps of type $\{3, 7\}$ that cover the Klein map of genus 3). These covers were critical to a complete determination of all Hurwitz surfaces (or equivalently, all regular maps of type $\{3, 7\}$) of genus 2 to 11905, achieved by the first author in [5, 6].

Suppose the finite group G acts as a group of automorphisms of a structure X of a particular type, such that the action is prescribed by a universal group $\mathcal{U}$, with G being a smooth quotient $\mathcal{U}/K$ of $\mathcal{U}$ by some torsion-free normal subgroup K. (For example, $\mathcal{U}$ could be the $(2, 3, 7)$ triangle group Δ, and K the fundamental group of some Hurwitz surface X for which $G = \mathrm{Aut}\, X$.) If Y is a cover of X admitting a group action of the same type, then there exists a normal subgroup L of $\mathcal{U}$ contained in K, with $\mathcal{U}/L$ being the corresponding group of automorphisms of Y. The group

$\mathcal{U}/L$ is then an extension of the covering group K/L by the given group $G = \mathcal{U}/K$.

In order to find all such covers that are abelian (for example), we need to find all possibilities for L such that K/L is abelian.

Typically, $\mathcal{U}$ is a finitely-presented group, and K has finite index in $\mathcal{U}$. In that case, a finite presentation for K can be found using Reidemeister-Schreier theory (see [20] or [25]), or by use of the `Rewrite` command in Magma [3]. Similarly, algebraic or computational techniques can be applied to find the actions by conjugation of the generators of $\mathcal{U}$ on the generators of K.

In the cases we will consider, K is a free group of finite rank (namely the dimension of the cycle space of the base graph), while in the case of Hurwitz surfaces, K is a surface-kernel group of genus g, generated by $2g$ elements $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ subject to a single defining relation $[a_1, b_1][a_2, b_2] \ldots [a_g, b_g] = 1$. In all such cases, $K/K' = K/[K, K]$ is a free abelian group of finite rank, say d, and the action of $\mathcal{U}$ by conjugation on the generators of K induces an action on the generators of K/K' which can be represented by $d \times d$ matrices with regard to some basis $\{w_1, w_2, \ldots, w_d\}$ for K/K'. This turns K/K' into a G-module of rank d over $\mathbb{Z}$ (where $G = \mathcal{U}/K$).

To find all finite covers with abelian covering group of exponent m, say, we can reduce the matrices mod m, or equivalently, consider the action of $\mathcal{U}$ by conjugation on the generators of $K/K'K^{(m)}$, where $K^{(m)}$ is the characteristic subgroup of K generated by the mth powers of all elements of K. The problem then reduces to finding all possibilities for a subgroup L of finite index in K such that L contains $K'K^{(m)}$ and L is normal in $\mathcal{U}$.

If $m = p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}$ is the prime-power factorisation of m (with p_i distinct primes), then the factor group K/L is a direct product of its Sylow subgroups, each of which is of the form K/Q_i where Q_i is a $\mathcal{U}$-invariant subgroup of K containing $K'K^{(m)}$ with index $|K\!:\!Q_i|$ dividing $p_i^{e_i}$. It follows that we need only find the $\mathcal{U}$-invariant subgroups of prime-power index in K/K', in order to find all finite abelian covers.

The key to this step of our approach is to take m as a prime-power, say $m = p^e$, and then view the group $K/K'K^{(m)}$ as being made up of e different 'layers'. For $0 \le j \le e$ we define $K_j = K'K^{p^j}$, the subgroup generated by K' and the (p^j)th powers of all elements of K. This is a characteristic subgroup of K and therefore normal in $\mathcal{U}$. The quotients $K_0/K_1, K_1/K_2, \ldots, K_{e-1}/K_e$ are what we view as the layers of $K/K'K^{(m)}$, from the top down. Each layer K_{j-1}/K_j is an elementary abelian p-group of rank d, generated by the cosets $K_j w_i$ for $1 \le i \le d$. These layers are illustrated in Figure 1.

Similarly, if L is any $\mathcal{U}$-invariant subgroup of K containing $K'K^{(m)}$ and with index $|K : L|$ a power of p, then we can define $L_j = L \cap K_j$ for $0 \le j \le e$, and view L as being made up of the layers $L_0/L_1, L_1/L_2, \ldots, L_{e-1}/L_e$, again from the top down. Each layer L_{j-1}/L_j of $L/K'K^{(m)}$ is a $\mathcal{U}$-invariant subgroup of the corresponding layer K_{j-1}/K_j of K. Since K_{j-1}/K_j is an elementary abelian p-group, possibilities for the layers L_{j-1}/L_j are relatively easy to find, in the same way that covers with elementary abelian covers are found.

Here it is helpful to consider the character table of the group $G = \mathcal{U}/K$, which

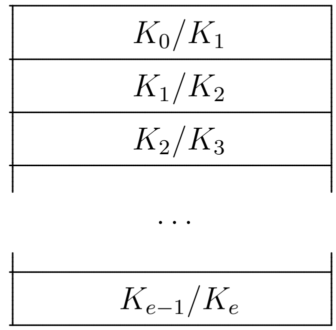

Figure 1: The layers of K

gives the degrees (and other details) of irreducible representations of G in characteristic 0. If the prime p does not divide the order of the group G, then these are also the degrees of the irreducible representations of G in characteristic p, and the other character values are helpful for finding all possibilities. (This can also be achieved using tools in Magma [3] for dealing with modules and submodules.)

Once we know the $\mathcal{U}$-invariant subgroups of the top layer K_0/K_1, including expressions for their generators in terms of the (images of the) generators w_i of K/K', we immediately have the same for other layers K_{j-1}/K_j for all $j > 1$. For example, if one such subgroup of the top layer is generated by the cosets K_1w_1 and $K_1w_2{w_3}^{-1}$, then each subsequent layer K_{j-1}/K_j has a $\mathcal{U}$-invariant subgroup generated by the cosets $K_jw_1^{p^{j-1}}$ and $K_jw_2^{p^{j-1}}({w_3}^{p^{j-1}})^{-1}$, since for any integer $q > 1$, conjugation of powers w_i^q of the w_i by generators of $\mathcal{U}$ is represented by the same matrices as conjugation of the w_i themselves.

What is more challenging is to find all possibilities for L by piecing together the possibilities for its layers. This, however, can be done by considering what happens for small values of e (and hence small values of $p^e = m$). For example, the $\mathcal{U}$-invariant subgroups of the top 'double-layer' group K_0/K_2 (of exponent p^2) tell us the possibilities for every double-layer. Also inspection of the generating sets for the $\mathcal{U}$-invariant subgroups can tell us possibilities for 'triple-layers', and so on.

We believe this 'layer' approach is new, and the examples in the next Section will show how it works in practice.

Finally, once all possibilities for L (and hence for the covering group K/L) have been found, we can determine additional information about the cover, such as uniqueness up to isomorphism, or the existence of additional symmetries, which occur when the kernel K is normal in some larger universal group $\mathcal{U}'$. For example, if $\mathcal{U}$ is the group G_1 from Section 2, and $G = \mathcal{U}/K$ is a 1-arc-regular group of automorphisms of the base graph X, then $\mathcal{U}/L$ is a 1-arc-regular group of automorphisms of the corresponding cover Y, and Y admits a 2-arc-regular group of automorphisms if and only if K is normal in the group $\mathcal{U}' = G_2^1$. Again, these things will be illustrated in the cases that follow.

4 Arc-transitive abelian covers of K_4

In this section, we classify all the arc-transitive abelian regular covering graphs of the complete graph K_4. We know that K_4 is 2-arc-regular, and belongs to the class 2^1. Its automorphism group is S_4 of order 24, and the only other arc-transitive group of automorphisms of K_4 is the subgroup A_4, of order 12, which acts regularly on the arcs of K_4.

Take the group $G_2^1 = \langle\, h, a, p \mid h^3 = a^2 = p^2 = (hp)^2 = (ap)^2 = 1 \,\rangle$. This group has two normal subgroups of index 24, both with quotient S_4, but these are interchanged by the outer automorphism of G_2^1 (induced by conjugation by an element of the larger group G_3) that takes the three generators h, a and p to h, ap and p respectively, and so without loss of generality we can take either one of them. We will take the one that is contained in the subgroup $G_1 = \langle h, a\rangle$; this is the unique normal subgroup N of index 12 in G_1 with $G_1/N \cong A_4$.

By Reidemeister-Schreier theory (see [20] or [25]), or by use of the `Rewrite` command in Magma, we find that the subgroup N is free of rank 3, on generators

$$w_1 = (ha)^3, \quad w_2 = (ah)^3 \quad \text{and} \quad w_3 = h^{-1}(ah)^3h.$$

Easy calculations show that the generators h, a and p act by conjugation as below:

$$\begin{array}{lll lll lll}
h^{-1}w_1h & = & w_2 & a^{-1}w_1a & = & w_2 & p^{-1}w_1p & = & w_2^{-1} \\
h^{-1}w_2h & = & w_3 & a^{-1}w_2a & = & w_1 & p^{-1}w_2p & = & w_1^{-1} \\
h^{-1}w_3h & = & w_1 & a^{-1}w_3a & = & w_1^{-1}w_3^{-1}w_2^{-1} & p^{-1}w_3p & = & w_3^{-1}.
\end{array}$$

Now take the quotient G_2^1/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ by the group $G_2^1/N \cong S_4$, and replace the generators h, a, p and all w_i by their images in this group. Also let K denote the subgroup N/N', and let G be G_1/N'. Then, in particular, G is an extension of $\mathbb{Z}^3$ by A_4.

By the above observations, we see that the generators h, a and p induce linear transformations of the free abelian group $K \cong \mathbb{Z}^3$ as follows:

$$h \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad p \mapsto \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These matrices generate a group isomorphic to S_4, with the first two generating a subgroup isomorphic to A_4. Note that the matrices of orders 3 and 2 representing h and a have traces 0 and -1, respectively.

Next, the character table of the group A_4 is as follows:

Element order	1	2	3	3
Class size	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	λ	λ^2
χ_3	1	1	λ^2	λ
χ_4	3	-1	0	0

where λ is a primitive cube root of 1.

By inspecting traces, we see that the character of the 3-dimensional representation of A_4 over $\mathbb{Q}$ associated with the above action of $G = \langle h, a\rangle$ on K is the character χ_4, which is irreducible. It follows that when we reduce by any prime k that does not divide $|A_4| = 12$, the corresponding action of A_4 on $K/K^{(k)} \cong \mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k$ is also irreducible, and hence the only non-trivial subgroup of $K/K^{(k)}$ that is normal in $G/K^{(k)}$ is $K/K^{(k)}$ itself.

The same argument holds for each 'layer' $K_i/K_{i+1} = K^{(k^i)}/K^{(k^{i+1})}$ of K, since this layer is generated by the cosets of $K^{(k^{i+1})}$ containing $w_1^{k^i}$, $w_2^{k^i}$ and $w_3^{k^i}$, and the effect of conjugation by each of h and a on these generators is given by the same matrices as above.

Similarly, when $k = 3$ we find that the action of G on $K/K^{(k)}$ is irreducible, because there are no subgroups of $K/K^{(3)} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ of order 3 or 9 that are normalized by both h and a. This is an easy exercise, verifiable with the help of Magma. (Note also that the mod 3 reductions of the characters χ_2 and χ_3 are both trivial.) Hence in particular, the same holds for the action of G on each layer K_i/K_{i+1}.

On the other hand, when $k = 2$ we find that the action of G on $K/K^{(k)}$ is reducible — indeed the subgroup of $K/K^{(2)}$ of order 4 generated by the cosets $K^{(2)}w_1w_2$ and $K^{(2)}w_1w_3$ normalized by both h and a — but there is no subgroup of order 2 with that property.

In fact, because the rank of K is small, we can also prove these (and more) things directly.

For any prime k, suppose $L/K^{(k)}$ is a non-trivial cyclic subgroup of $K/K^{(k)}$ that is normalized by both h and a, and suppose $L/K^{(k)}$ is generated by the coset $K^{(k)}x$ where $x = w_1^{\alpha}w_2^{\beta}w_3^{\gamma} \in K$. Then L contains $x^h = w_1^{\gamma}w_2^{\alpha}w_3^{\beta}$, so $(\gamma, \alpha, \beta) \equiv \lambda(\alpha, \beta, \gamma)$ mod k for some $\lambda \in \mathbb{Z}_k$, and then since $\alpha \equiv \lambda\beta \equiv \lambda^2\gamma \equiv \lambda^3\alpha$ mod k, each of α, β and γ must be non-zero mod k, and $\lambda^3 \equiv 1$ mod k. Next, L contains x^a, and so contains $w_1^{\beta-\gamma}w_2^{\alpha-\gamma}w_3^{-\gamma}$ and hence also $w_1^{\gamma-\beta}w_2^{\gamma-\alpha}w_3^{\gamma}$, and then because $\gamma \not\equiv 0$ mod k, it follows that $\gamma \equiv \alpha + \beta$ mod k. Similarly, $\alpha \equiv \beta + \gamma$ and $\beta \equiv \alpha + \gamma$ mod k, and then adding these congruences gives $\alpha + \beta \equiv \alpha + \beta + 2\gamma$ mod k, so $2\gamma \equiv 0$ mod k, which implies that $k = 2$. But then $\lambda = 1$, in which case $\alpha \equiv \beta \equiv \gamma \equiv 1$ mod k, and therefore $\gamma \not\equiv \alpha + \beta$ mod k, a contradiction. Hence no such cyclic subgroup exists.

On the other hand, let $L/K^{(k)}$ be a subgroup of $K/K^{(k)}$ isomorphic to $\mathbb{Z}_k \oplus \mathbb{Z}_k$ that is normalized by both h and a. Then L must contain all elements of the form $w_1^{\sigma}w_2^{\tau}w_3^{\mu}$, where (σ, τ, μ) lies in a 2-dimensional subspace of $\mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k$ (as a vector space over $\mathbb{Z}_k$). The latter is the orthogonal complement of a unique 1-dimensional subspace, generated say by (α, β, γ). Since $L/K^{(k)}$ is normalized by h, we find that every such (σ, τ, μ) is orthogonal also to (γ, α, β) and (β, γ, α), and so as above, we find that $\alpha \equiv \lambda\beta \equiv \lambda^2\gamma \bmod k$ for some primitive cube root λ of 1 in $\mathbb{Z}_k$. Similarly, because $L/K^{(k)}$ is normalized by a, we find that for any such (σ, τ, μ), also $(\tau-\mu, \sigma-\mu, -\mu)$ is orthogonal to (α, β, γ), and hence so is $(\tau-\mu, \sigma-\mu, -\mu)+(\sigma, \tau, \mu) = (\sigma+\tau-\mu, \sigma+\tau-\mu, 0)$, giving $0 \equiv (\sigma+\tau-\mu)(\alpha+\beta) = (\sigma+\tau-\mu)(1+\lambda)\beta \bmod k$.

Now if $\lambda \not\equiv -1 \bmod k$, then $\sigma+\tau-\mu \equiv 0 \bmod k$ for all such (σ, τ, μ), in which case we can take $(\alpha, \beta, \gamma) \equiv (1, 1, -1) \bmod k$, but that is impossible since we need $\alpha \equiv \lambda\beta \equiv \lambda^2\gamma \bmod k$. Thus $\lambda \equiv -1 \bmod k$. Moreover, since $-1 \equiv \lambda^3 \equiv 1 \bmod k$, again we get $k = 2$, and $\lambda = 1$ and $\alpha \equiv \beta \equiv \gamma \equiv 1 \bmod k$. Hence there is only one G-invariant subgroup of rank 2 in $K/K^{(2)}$, namely the subgroup generated by $K^{(2)}w_1w_2$ and $K^{(2)}w_1w_3$, and there are no G-invariant subgroups of rank 2 in $K/K^{(k)}$ when k is an odd prime.

Note that the above arguments hold also for each layer K_i/K_{i+1} of K.

So now suppose $m = k^e$ is any prime-power greater than 1, and suppose $L/K^{(m)}$ is any non-trivial normal subgroup of $G/K^{(m)}$ contained in $K/K^{(m)}$.

If k is odd, then by the above arguments, we know that each layer $L_i/L_{i+1} = (L\cap K_i)/(L\cap K_{i+1})$ of L (for $1 \le i \le e$) is either trivial or isomorphic to $\mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k$. Let t be the smallest non-negative integer for which L_t/L_{t+1} is non-trivial. Then L_t must contain the cosets of $K^{(k^{t+1})}$ represented by each of $w_1^{k^t}, w_2^{k^t}$ and $w_3^{k^t}$, and hence also L_j must contain the cosets represented by $w_1^{k^j}, w_2^{k^j}$ and $w_3^{k^j}$, for $t \le j < e$, and it follows by an easy induction that $L/K^{(m)} \cong \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$, where $d = k^{e-t}$.

On the other hand, if $k = 2$, then it is possible that the first non-trivial layer L_t/L_{t+1} of L is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (rather than $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$). To explain what happens in that case, we will assume that $t = 0$, but note that the general case (where t may be greater than 0) is similar.

In the case $t = 0$, the group $L/K^{(k)}$ is generated by the cosets $K^{(k)}w_1w_2$ and $K^{(k)}w_1w_3$, and so L must contain the elements $x = w_1w_2^{\beta}w_3^{\gamma}$ and $y = w_1w_2^{\tau}w_3^{\mu}$ for some odd integers β and μ and even integers γ and τ. Then since L is normalized by h and a, it contains each of x^h, x^a and x^{ha}, and therefore contains $w_1^{\gamma}w_2w_3^{\beta}$, $w_1^{\beta-\gamma}w_2^{1-\gamma}w_3^{-\gamma}$ and $w_1^{1-\beta}w_2^{\gamma-\beta}w_3^{-\beta}$. Hence in particular, L contains the product $xx^hx^ax^{ha} = w_1^{1+\gamma+\beta-\gamma+1-\beta}w_2^{\beta+1+1-\gamma+\gamma-\beta} = w_1^2w_2^2$. Conjugation by h then shows that L contains $w_1^2w_3^2$ and $w_2^2w_3^2$, and by multiplying $w_1^2w_2^2$ by the inverse of one of these two, we find that L contains also $w_2^2w_3^{-2}$ and $w_1^2w_3^{-2}$, and hence L contains each of w_1^4, w_2^4 and w_3^4. Thus $L/K^{(m)} \cong \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_{m/2}$ or $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_{m/4}$. In particular, if L contains one (and hence all) of w_1^2, w_2^2 and w_3^2, then $L/K^{(m)} \cong \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_{m/2}$ and is generated by the cosets of $K^{(m)}$ represented by w_1w_2, w_1w_3 and w_3^2, while otherwise $L/K^{(m)} \cong \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_{m/4}$ and is generated by the cosets of $K^{(m)}$ represented by $w_1w_2^{-1}$, $w_1w_3^{-1}$ and $w_1^2w_2^2$ (or equivalently, by $w_1w_2^{-1}$, $w_1w_3^{-1}$ and w_1^4).

In the general case, if L contains one (and hence all) of $w_1^{2^{t+1}}$, $w_2^{2^{t+1}}$ and $w_3^{2^{t+1}}$, then $L/K^{(m)} \cong \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{d/2}$ where $d = 2^{e-t}$, and $L/K^{(m)}$ is generated by the cosets of $K^{(m)}$ represented by $w_1^{2^t}w_2^{2^t}$, $w_1^{2^t}w_3^{2^t}$ and $w_3^{2^{t+1}}$; otherwise $L/K^{(m)} \cong \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{d/4}$ and is generated by the cosets of $K^{(m)}$ represented by $w_1^{2^t}w_2^{-2^t}$, $w_1^{2^t}w_3^{-2^t}$ and $w_1^{2^{t+2}}$. The covering group K/L is then isomorphic to $\mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^{t+1}}$ or $\mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^{t+2}}$, respectively. These two possibilities are illustrated in Figure 2 for the case $t = 3$.

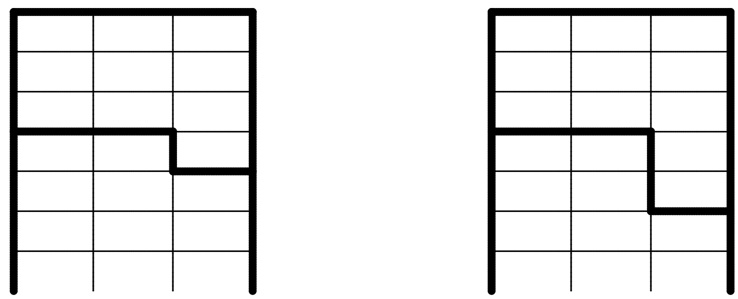

Figure 2: The two possibilities for non-homocyclic covers of 2-power exponent

Note that if we let T_0, T_1 and T_2 denote the three possibilities for a G-invariant subgroup of $K/K^{(2)}$ (of ranks 0, 2 and 3 respectively), then we can think of these two non-homocyclic possibilities for $L/K^{(m)}$ as a 'tower' of copies of T_2 with one or two copies of T_1 on the top. This way of looking at the structure of $L/K^{(m)}$ will be developed further in the later examples (in Sections 5, 6 and 7).

In summary, Table 4.1 below provides all possibilities for a normal subgroup L of G contained with finite prime-power index in K:

Index $\|K:L\|$	Generating set for L	Quotient K/L
$\ell^3 = k^{3t}$, any prime k	$\{w_1^\ell,\ w_2^\ell,\ w_3^\ell\}$	$\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell$
$2\ell^3 = 2^{3t+1}$	$\{w_1^\ell w_2^\ell,\ w_1^\ell w_3^\ell,\ w_3^{2\ell}\}$	$\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_{2\ell}$
$4\ell^3 = 2^{3t+2}$	$\{w_1^\ell w_2^{-\ell},\ w_1^\ell w_3^{-\ell},\ w_1^{2\ell} w_2^{2\ell}\}$	$\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_{4\ell}$

Table 4.1: Possibilities for each G_1-invariant subgroup L of K when $G_1/K \cong A_4$

Finally, it is easy to see that in all of the above cases, the subgroup L is normal not only in $G = G_1/N'$ but also in the larger group G_2^1/N', since conjugation by the additional generator p has the following effects on the relevant generators:

$$\begin{aligned} w_1^i w_2^i &\mapsto w_2^{-i} w_1^{-i} = (w_1^i w_2^i)^{-1}, & w_1^i w_2^{-i} &\mapsto w_2^{-i} w_1^{i} = w_1^i w_2^{-i}, \\ w_1^i w_3^i &\mapsto w_2^{-i} w_3^{-i} = (w_1^i w_2^i)(w_1^i w_3^i)^{-1}, & w_1^i w_3^{-i} &\mapsto w_2^{-i} w_3^{i} = (w_1^i w_2^{-i})(w_1^i w_3^{-i})^{-1}, \\ w_3^j &\mapsto (w_3^j)^{-1}, & w_1^j w_2^j &\mapsto w_2^{-j} w_1^{-j} = (w_1^j w_2^j)^{-1}. \end{aligned}$$

Hence in particular, each of the resulting covers is 2-arc-transitive.

In fact, in each case the subgroup K/L is characteristic in the quotient G_2^1/L. This is clear for $k \notin \{2, 3\}$ by Sylow theory, while for $k = 3$ it is not difficult to see that K/L is the largest abelian normal 3-subgroup of G_2^1/L, and for $k = 2$, there is no other abelian normal 2-subgroup of G_2^1/L that is isomorphic to K/L (even when $|K/L|$ is small). It follows that each L is not G_3-invariant, for if it were, then G_2^1/L would be invariant under the outer automorphism of G_2^1 that takes h, a and p to h, ap and p (respectively), but then that automorphism would have to preserve the characteristic subgroup K/L and hence preserve K, which we know is not the case (since K_4 is 2-arc-regular). Hence none of these covers can be 3-arc-regular.

Finally, by [12, Theorem 3] none of these covers can be 4- or 5-arc-transitive, since each admits a 2-arc-regular group of automorphisms.

Thus each of these covers is 2-arc-regular, and we have the following:

Theorem 4.1 *Let m be any prime-power. If m is odd, then the complete graph K_4 has just one arc-transitive regular cover with abelian covering group of exponent m, namely a 2-arc-regular cover of type 2^1 with covering group $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$. On the other hand, if $m = 2^e$ for some $e \geq 2$, then K_4 has exactly three such covers, which are 2-arc-regular covers of type 2^1 with covering groups $\mathbb{Z}_{m/4} \oplus \mathbb{Z}_{m/4} \oplus \mathbb{Z}_m$, $\mathbb{Z}_{m/2} \oplus \mathbb{Z}_{m/2} \oplus \mathbb{Z}_m$ and $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, while if $m = 2$ then there are exactly two such covers, which are 2-arc-regular covers of type 2^1 with covering groups $\mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In particular, every arc-transitive abelian regular cover of the complete graph K_4 is 2-arc-regular.*

5 Arc-transitive abelian covers of $K_{3,3}$

In this section, we classify all the arc-transitive covering graphs of the complete bipartite graph $K_{3,3}$, with abelian covering group. We know that $K_{3,3}$ is 3-arc-regular, so belongs to the class 3. Its automorphism group is the wreath product $S_3 \wr S_2$, of order 72, and this contains unique arc-transitive subgroups of type 1, 2^1 and 2^2, having orders 18, 36 and 36 respectively (see [10, §4.5]).

In particular, two of these subgroups are minimal arc-transitive groups of automorphisms of $K_{3,3}$: one is the group $A_3 \wr S_2$ (of order 18), which acts regularly on the arcs of $K_{3,3}$, while the other is a subgroup of order 36 which acts regularly on the 2-arcs of $K_{3,3}$ with an action of type 2^2. We will investigate abelian regular covers of $K_{3,3}$ using the former subgroup, and then later consider what happens with the latter subgroup.

Take $G_3 = \langle\, h, a, p, q \mid h^3 = a^2 = p^2 = q^2 = [p, q] = [h, p] = (hq)^2 = apaq = 1\,\rangle$, let $G = G_1 = \langle h, a\rangle$, and let N be the unique normal subgroup of index 72 in G_3 (and index 18 in G_1), with quotients $G_3/N \cong S_3 \wr S_2$ and $G_1/N \cong A_3 \wr S_2$. Using Reidemeister-Schreier theory or the `Rewrite` command in Magma, we find that the subgroup N is free of rank 4, on generators

$$w_1 = hahah^{-1}ah^{-1}a, \quad w_2 = h^{-1}ahahah^{-1}a,$$
$$w_3 = hah^{-1}ah^{-1}aha, \quad w_4 = h^{-1}ah^{-1}ahaha,$$

and easy calculations show that the generators h, a, p and q act by conjugation as below:

$$\begin{array}{rclrclrclrcl}
h^{-1}w_1h &=& w_2^{-1} & a^{-1}w_1a &=& w_1^{-1} & p^{-1}w_1p &=& w_3 & q^{-1}w_1q &=& w_2\\
h^{-1}w_2h &=& w_1w_2^{-1} & a^{-1}w_2a &=& w_3^{-1} & p^{-1}w_2p &=& w_4 & q^{-1}w_2q &=& w_1\\
h^{-1}w_3h &=& w_4^{-1} & a^{-1}w_3a &=& w_2^{-1} & p^{-1}w_3p &=& w_1 & q^{-1}w_3q &=& w_4\\
h^{-1}w_4h &=& w_3w_4^{-1} & a^{-1}w_4a &=& w_4^{-1} & p^{-1}w_4p &=& w_2 & q^{-1}w_4q &=& w_3.
\end{array}$$

Now take the quotient G_3/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^4$ by the group $G_3/N \cong S_3 \wr S_2$, and replace the generators h, a, p, q and all w_i by their images in G_3/N'. Also let K denote the subgroup N/N', and let $G = G_1/N'$, so that G is an extension of K by $G/K \cong A_3 \wr S_2$.

By the above observations, the generators h, a, p and q induce linear transformations of the free abelian group $K \cong \mathbb{Z}^4$ as follows:

$$h \mapsto \begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$p \mapsto \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q \mapsto \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that the matrices of orders 3 and 2 representing h and a both have trace -2, while the matrices of orders 6, 3 and 3 representing ha, $[h, a]$ and $(ha)^2$ (which is central in the subgroup generated by matrices representing h and a) all have trace 1.

Next, the character table of the group $A_3 \wr S_2$ is as follows:

Element order	1	2	3	3	3	3	3	6	6
Class size	1	3	1	1	2	2	2	3	3
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	1	-1	-1
χ_3	1	-1	λ	λ^2	1	λ	λ^2	$-\lambda$	$-\lambda^2$
χ_4	1	-1	λ^2	λ	1	λ^2	λ	$-\lambda^2$	$-\lambda$
χ_5	1	1	λ	λ^2	1	λ	λ^2	λ	λ^2
χ_6	1	1	λ^2	λ	1	λ^2	λ	λ^2	λ
χ_7	2	0	2	2	-1	-1	-1	0	0
χ_8	2	0	2λ	2λ	-1	$-\lambda$	$-\lambda^2$	0	0
χ_9	2	0	$2\lambda^2$	$2\lambda^2$	-1	$-\lambda^2$	$-\lambda$	0	0

where λ is a primitive cube root of 1.

By inspecting traces, we see that the character of the 4-dimensional representation of $A_3 \wr S_2$ over $\mathbb{Q}$ associated with the above action of $G = \langle h, a \rangle$ on K is $\chi_3 + \chi_4 + \chi_7$, which is reducible to the sum of $\chi_3+\chi_4$ and χ_7, which are characters of two irreducible 2-dimensional representations over the rational field $\mathbb{Q}$.

It follows that for every prime $k \notin \{2,3\}$, the group $K/K^{(k)} \cong (\mathbb{Z}_k)^4$ is the direct sum of two G-invariant subgroups of rank 2, and if $\mathbb{Z}_k$ contains a non-trivial cube root of 1, then one of these is the direct sum of two G-invariant cyclic subgroups and the other is irreducible, while otherwise both of them are irreducible.

In fact, the rank 2 subgroup of K generated by $x = w_1w_3^{-1}w_4$ and $y = w_2w_3^{-1}$ is normal in G, with $x^h = y^{-1}$, $y^h = xy^{-1}$, $x^a = x^{-1}y$ and $y^a = y^{-1}$. Similarly, the rank 2 subgroup generated by $u = w_1w_4^{-1}$ and $v = w_2w_3w_4^{-1}$ is normal in G, with $u^h = v^{-1}$, $v^h = uv^{-1}$, $u^a = u^{-1}$ and $v^a = v^{-1}$. In the quotient $G/K^{(k)}$, the image of the rank 2 subgroup $\langle x,y\rangle$ has no non-trivial G-invariant cyclic subgroup, and hence is irreducible. If $k \equiv 2$ mod 3, then the image of $\langle u,v\rangle$ is also irreducible, while if $k \equiv 1$ mod 3 and λ is a primitive cube root of 1 mod k, then the image of $\langle u,v\rangle$ is the direct sum of G-invariant subgroups generated by the images of each of $z_\lambda = w_1w_2^\lambda w_3^\lambda w_4^{\lambda^2}$ and $z_{\lambda^2} = w_1w_2^{\lambda^2}w_3^{\lambda^2}w_4^\lambda$, while if $k \equiv 2$ mod 3 then it is irreducible.

For $k = 3$, the group $K/K^{(k)}$ is still has two G-invariant subgroups of rank 2, one generated by the images of $x = w_1w_3^{-1}w_4$ and $y = w_2w_3^{-1}$ and the other generated by the images of $u = w_1w_4^{-1}$ and $v = w_2w_3w_4^{-1}$; but in this case they have non-trivial intersection, namely the cyclic (and G-invariant) subgroup generated by $K^{(k)}w_1w_2w_3w_4$, which equals both $K^{(k)}xy$ and $K^{(k)}uv$. In fact the only non-trivial proper G-invariant subgroups of $K/K^{(3)}$ are these two subgroups of rank 2, their intersection (of rank 1), and the rank 3 subgroup that they generate together.

For $k = 2$, the group $K/K^{(k)}$ is again the direct sum of two G-invariant subgroups of rank 2, just as in the case for any larger prime $k \equiv 2$ mod 3.

The above observations were made for the 'top' layer $K_0/K_1 = K/K^{(k)}$ of K, and give all the elementary abelian regular covers. But also the analogous observations hold for every other layer K_i/K_{i+1} of K (with $i \geq 1$).

So now suppose $m = k^e$ is any prime-power greater than 1, and suppose $L/K^{(m)}$ is any non-trivial normal subgroup of $G/K^{(m)}$ contained in $K/K^{(m)}$.

If $k \equiv 2$ mod 3, then each layer L_i/L_{i+1} of L is either trivial or isomorphic to $\mathbb{Z}_k \oplus \mathbb{Z}_k$ or $\mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k$. It then follows from the above observations that there exist divisors c and d of $m = k^e$ such that $L/K^{(m)} \cong \mathbb{Z}_c \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$, with L being generated by $(w_1w_4^{-1})^{m/c}$, $(w_2w_3w_4^{-1})^{m/c}$, $(w_1w_3^{-1}w_4)^{m/d}$ and $(w_2w_3^{-1})^{m/d}$.

On the other hand, if $k \equiv 1$ mod 3, and λ is a primitive cube root of 1 mod m, then there exist divisors b, c and d of m such that $L/K^{(m)} \cong \mathbb{Z}_b \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$, with L generated by $(w_1w_2^\lambda w_3^\lambda w_4^{\lambda^2})^{m/b}$, $(w_1w_2^{\lambda^2}w_3^{\lambda^2}w_4^\lambda)^{m/c}$, $(w_1w_3^{-1}w_4)^{m/d}$ and $(w_2w_3^{-1})^{m/d}$.

The case $k = 3$ is not quite so straightforward. In this case, each layer can have rank 0, 1, 2, 3 or 4, depending on the layers above it. To see exactly what happens, it is helpful to consider the case $m = 3^2 = 9$. For notational convenience, let $x = w_1w_3^{-1}w_4$, $y = w_2w_3^{-1}$, $u = w_1w_4^{-1}$, $v = w_2w_3w_4^{-1}$ and $z = w_1w_2w_3w_4$, and let $\overline{g}$ denote the coset $K^{(9)}g$, when $g \in K$. Then an easy exercise (using MAGMA if necessary) shows that $K/K^{(9)}$ contains exactly 24 subgroups that are normal in

$G/K^{(9)}$, and these may be summarised as follows:

- the group $K/K^{(9)} \cong \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$ itself, generated by all the $\overline{w_i}$;
- one subgroup isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$, generated by $\overline{x}, \overline{y}, \overline{u}$ (or $\overline{v}$) and $\overline{w_4}^3$;
- two subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, one generated by $\overline{x}, \overline{y}$ and all $\overline{w_i}^3$, and the other by $\overline{u}, \overline{v}$ and all $\overline{w_i}^3$;
- one subgroup isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, generated by $\overline{z}$ and all $\overline{w_i}^3$;
- four subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$, generated by $\{\overline{x}, \overline{y}, \overline{u}^3\}$, $\{\overline{u}, \overline{v}, \overline{y}^3\}$, $\{\overline{u}\,\overline{w_1}^3, \overline{v}\,\overline{w_2}^3, \overline{y}^3\}$ and $\{\overline{u}\,\overline{w_2}^3, \overline{v}\,\overline{w_1}^3, \overline{y}^3\}$;
- three subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, generated by $\overline{x}^3$ and $\overline{y}^3$ plus one of $\overline{z}$, $\overline{z}\,\overline{w_4}^3$ and $\overline{z}\,\overline{w_4}^6$, respectively;
- two subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9$, generated by $\{\overline{x}, \overline{y}\}$ and $\{\overline{u}, \overline{v}\}$;
- four subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_3$, generated by $\{\overline{z}\,\overline{w_3}^6, \overline{y}^3\}$, $\{\overline{z}\,\overline{w_4}^3, \overline{y}^3\}$, $\{\overline{z}\,\overline{w_3}^3\,\overline{w_4}^6, \overline{y}^3\}$ and $\{\overline{z}\,\overline{w_4}^6, \overline{u}^3\}$;
- six G-invariant subgroups of exponent 1 or 3 lying in the second layer $K^{(3)}/K^{(9)}$, isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3$ and $\mathbb{Z}_1$.

Note that if we let T_0, T_1, T_2, T_3, T_4 and T_5 denote the six possibilities for a G-invariant subgroup of $K/K^{(3)}$ (of ranks 0, 1, 2, 2, 3 and 4 respectively), with T_2 generated by the images of x and y, and T_3 generated by the images of u and v, then we can represent each of the above subgroup types as a pair (T_i, T_j), where T_i indicates the first layer L_0/L_1 of the subgroup L, and T_j denotes the second layer L_1/L_2. Then in order, the pairs that occur are as follows:

- (T_5, T_5) once, • (T_4, T_5) once, • (T_2, T_5) and (T_3, T_5) once each, • (T_1, T_5) once,
- (T_2, T_4) once and (T_3, T_4) three times, • (T_1, T_4) three times,
- (T_2, T_2) and (T_3, T_3) once each, • (T_1, T_2) three times and (T_1, T_3) once,
- (T_0, T_5), (T_0, T_4), (T_0, T_3), (T_0, T_2), (T_0, T_1) and (T_0, T_0), once each.

The same argument shows that each 'double-layer' section K_i/K_{i+2} of K has exactly 24 G-invariant subgroups, analogous to those in the summary lists above.

As a consequence of these observations, we can draw three important conclusions.

First, for $m = 3^e$ the only G-invariant cyclic subgroups of $K/K^{(m)}$ have orders 1 and 3, because there is no subgroup in the above list isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$. In fact, if any layer L_i/L_{i+1} of a G-invariant subgroup L of K is non-trivial, then it contains the coset $L_{i+1}(w_1w_2w_3w_4)^{3^i}$, and so contains its image under conjugation by h, namely the coset $L_{i+1}(w_1w_2^{-2}w_3w_4^{-2})^{3^i}$, and hence contains also $L_{i+1}(w_2w_4)^{3^{i+1}}$, in which case the layer L_{i+1}/L_{i+2} has rank 2 or more. In other words, if one layer is T_1, then the next must be T_2, T_3, T_4 or T_5.

Second, the only G-invariant proper subgroup L of K with cyclic quotient K/L of 3-power order is the subgroup generated by $\overline{x}, \overline{y}, \overline{u}$ and $\overline{w_4}^3$, with quotient $K/L \cong C_3$. Hence if the top layer L_0/L_1 of a G-invariant subgroup L of K is T_4 (of rank 3), then the next layer L_1/L_2 is T_5 (of rank 4). Furthermore, the same argument applied to deeper layers shows that if any layer L_i/L_{i+1} of a G-invariant subgroup L of K is T_4 (of rank 3), then the next layer L_{i+1}/L_{i+2} must be T_5 (of rank 4).

Third, if any layer L_i/L_{i+1} of a G-invariant subgroup L of K has rank 2, then the next layer L_{i+1}/L_{i+2} can have rank 2, 3 or 4 (and if it has rank 3, then the subsequent layer L_{i+2}/L_{i+3} must have rank 4). In fact, the possible pairs for two successive layers in this case are (T_2, T_2), (T_2, T_4), (T_2, T_5), (T_3, T_3), (T_3, T_4) and (T_3, T_5), and then the possible triples for three successive layers are (T_2, T_2, T_2), (T_2, T_2, T_4), (T_2, T_2, T_5), (T_2, T_4, T_5), (T_2, T_5, T_5), (T_3, T_3, T_3), (T_3, T_3, T_4), (T_3, T_3, T_5), (T_3, T_4, T_5) and (T_3, T_5, T_5).

We can now put these observations together to find all possibilities for a normal subgroup L of $G = \langle h, a \rangle$ contained in K with index $|K : L|$ being a power $m = k^e$ of a prime k.

When $k = 3$, the layers of any such L must consist of (say) e_0 copies of T_0 (where $e_0 \geq 0$), followed by e_1 copies of T_1 (where $e_1 = 0$ or 1), followed by e_2 copies of either T_2 or T_3, followed by e_3 copies of T_4 (where $e_3 = 0$ or 1), followed by e_4 copies of T_5 (where $e_4 \geq 0$), with $e_0 + e_1 + e_2 + e_3 + e_4 = e$. Moreover, any such combination determines a unique L, except in the cases where a pair of successive layers is of the form (T_1, T_2), (T_1, T_4) or (T_3, T_4), where there are exactly three such L.

All the possibilities for L are listed in Table 5.1.

Index $\lvert K:L \rvert$	Generating set for L	Quotient K/L
$d^4 = k^{4t}$, for any k	$\{w_1^d,\ w_2^d,\ w_3^d,\ w_4^d\}$	$\mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$
$c^2d^2 = k^{2s}k^{2t}$, with $s < t$, for any $k \neq 3$	$\{u^c,\ v^c,\ x^d,\ y^d\}$ or $\{x^c,\ y^c,\ u^d,\ v^d\}$	$\mathbb{Z}_c \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$
$bcd^2 = k^r k^s k^{2t}$, with $r < s$, for any $k \equiv 1 \bmod 3$	$\{{z_\lambda}^b,\ {z_{\lambda^2}}^c,\ x^d,\ y^d\}$ or $\{{z_{\lambda^2}}^b,\ {z_\lambda}^c,\ x^d,\ y^d\}$	$\mathbb{Z}_b \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$
$3d^4 = 3^{4t+1}$	$\{x^d,\ y^d,\ u^d,\ w_4^{3d}\}$	$\mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{3d}$
$27d^4 = 3^{4t+3}$	$\{z^d,\ w_2^{3d},\ w_3^{3d},\ w_4^{3d}\}$	$\mathbb{Z}_d \oplus \mathbb{Z}_{3d} \oplus \mathbb{Z}_{3d} \oplus \mathbb{Z}_{3d}$
$81d^4 = 3^{4t+3}$	$\{z^d,\ y^{3d},\ u^{3d},\ w_4^{9d}\}$, or $\{z^d w_4^{3d},\ y^{3d},\ u^{3d},\ w_4^{9d}\}$ or $\{z^d w_4^{-3d},\ y^{3d},\ u^{3d},\ w_4^{9d}\}$	$\mathbb{Z}_d \oplus \mathbb{Z}_{3d} \oplus \mathbb{Z}_{3d} \oplus \mathbb{Z}_{9d}$
$c^2d^2 = 3^{2s}3^{2t}$, with $s < t$	$\{x^c,\ y^c,\ w_3^d,\ w_4^d\}$ or $\{u^c,\ v^c,\ w_3^d,\ w_4^d\}$	$\mathbb{Z}_c \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$
$3c^2d^2 = 3^{2s+1}3^{2t}$, with $s+1 < t$	$\{z^c w_4^{-3c},\ u^{3c},\ y^d,\ w_4^d\}$ or $\{z^c w_3^{-3c},\ y^{3c},\ u^d,\ w_4^d\}$ or $\{z^c w_3^{\frac{d}{3}-3c} w_4^{\frac{d}{3}},\ y^{3c},\ u^d,\ w_4^d\}$ or $\{z^c w_3^{-\frac{d}{3}-3c} w_4^{-\frac{d}{3}},\ y^{3c},\ u^d,\ w_4^d\}$	$\mathbb{Z}_c \oplus \mathbb{Z}_{3c} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$
$3c^2d^2 = 3^{2s}3^{2t+1}$, with $s+1 < t$	$\{x^c,\ y^c,\ u^d,\ w_4^{3d}\}$ or $\{u^c,\ v^c,\ y^d,\ w_4^{3d}\}$ or $\{u^c w_1^d,\ v^c w_2^d,\ y^d,\ w_4^{3d}\}$ or $\{u^c w_2^d,\ v^c w_1^d,\ y^d,\ w_4^{3d}\}$	$\mathbb{Z}_c \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{3d}$
$9c^2d^2 = 3^{2s+1}3^{2t+1}$, with $s+2 < t$	$\{z^c w_3^{-3c},\ y^{3c},\ u^d,\ w_4^{3d}\}$ or $\{z^c w_3^{d-3c},\ y^{3c},\ u^d,\ w_4^{3d}\}$ or $\{z^c w_3^{-d-3c},\ y^{3c},\ u^d,\ w_4^{3d}\}$ or $\{z^c w_4^{-3c},\ u^{3c},\ y^d,\ w_4^{3d}\}$ or $\{z^c w_4^{d-3c},\ u^{3c},\ y^d,\ w_4^{3d}\}$ or $\{z^c w_4^{-d-3c},\ u^{3c},\ y^d,\ w_4^{3d}\}$	$\mathbb{Z}_c \oplus \mathbb{Z}_{3c} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{3d}$

Table 5.1: Possibilities for each G_1-invariant subgroup L of K when $G_1/K \cong A_3 \wr S_2$

Next, we consider which of the G_1-invariant subgroups of $K/K^{(m)}$ are normalized by the additional generators p, q and pq of the larger groups G_2^1, G_2^2 and G_3.

Note that conjugation by pq (which lies in G_2^1/N but not G_1/N) takes

$$x = w_1 w_3^{-1} w_4 \ \mapsto\ w_4 w_2^{-1} w_1 = xy^{-1}, \qquad y = w_2 w_3^{-1} \ \mapsto\ w_3 w_2^{-1} = y^{-1},$$
$$u = w_1 w_4^{-1} \ \mapsto\ w_4 w_1^{-1} = u^{-1}, \qquad v = w_2 w_3 w_4^{-1} \ \mapsto\ w_3 w_2 w_1^{-1} = u^{-1} v,$$

while if λ is a primitive cube root of 1 in $\mathbb{Z}_m$, then modulo $K^{(m)}$, also conjugation by pq takes

$$z_\lambda = w_1 w_2^\lambda w_3^\lambda w_4^{\lambda^2} \;\mapsto\; w_4 w_3^\lambda w_2^\lambda w_1^{\lambda^2} = (w_1 w_2^{\lambda^2} w_3^{\lambda^2} w_4^\lambda)^{\lambda^2} = z_{\lambda^2}^{\lambda^2}.$$

It follows that in each layer of $K/K^{(m)}$, the G_1-invariant subgroups of rank 2 that we encountered above are invariant under conjugation by pq, while those of rank 1 are not, except when $k = 3$. Hence in particular, for $k \neq 3$, every G_1-invariant subgroup of $K/K^{(m)}$ for which each layer has rank 0, 2 or 4 is G_2^1-invariant, but those for which some layer has rank 1 or 3 are not. In other words, those in rows 1 and 2 of Table 5.1 are G_2^1-invariant, while those in row 3 are not.

In the exceptional case ($k = 3$), again some more careful attention is needed, but in fact it is easy to check that all G_1-invariant subgroups of $K/K^{(m)}$ are G_2^1-invariant apart from the 3rd and 4th subgroups in each of rows 8 and 9 of Table 5.1.

Similarly, conjugation by p takes

$$x = w_1 w_3^{-1} w_4 \;\mapsto\; w_3 w_1^{-1} w_2 = u^{-1}v, \qquad y = w_2 w_3^{-1} \;\mapsto\; w_4 w_1^{-1} = u^{-1},$$
$$u = w_1 w_4^{-1} \;\mapsto\; w_3 w_2^{-1} = y^{-1}, \qquad v = w_2 w_3 w_4^{-1} \;\mapsto\; w_4 w_1 w_2^{-1} = xy^{-1},$$

and hence interchanges the two rank 2 subgroups $\langle x, y\rangle$ and $\langle u, v\rangle$ of K.

It follows that when $k \neq 3$, the only G_3-invariant subgroups of $K/K^{(m)}$ are the homocyclic subgroups $\mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$ (for d dividing m). When $k = 3$, the only G_2^1-invariant subgroups of $K/K^{(m)}$ that are also G_3-invariant are those in rows 1, 4 and 5 and the first of the subgroups in row 6 of Table 5.1; in all other cases, conjugation by p takes the subgroup to another of the G_2^1-invariant subgroups from the same row.

By [12, Theorem 3], none of the covers obtained from the subgroups that are G_2^1- or G_3-invariant can be 4- or 5-arc-transitive, since each admits a 2- or 3-arc-regular group of automorphisms. Similarly, if the subgroup L is G_1-invariant but not G_2^1-invariant, then the cover admits a 1-arc-regular group of automorphisms, and cannot be 3-arc-regular by [12, Proposition 26] or [10, Proposition 2.3].

Next, suppose the subgroup L is G_1-invariant but not G_2^1-invariant, and the cover is 4-arc-transitive. In that case, the cover has both a 1-regular and a 4-arc-regular group of automorphisms, and so by [12, Proposition 29] or [10, Proposition 3.2], the cover must be a regular cover of the Heawood graph. Hence the quotient G_1/L must have a normal subgroup J/L of index $336/8 = 42$ in G_1/L, with $G_1/J \cong (G_1/L)/(J/L)$ isomorphic to an extension of C_7 by C_6, and J normal in the larger group G_4^1. In particular, $|G_1/L|$ is divisible by 7, and then since $|G_1/K| = |A_3 \wr S_2| = 18$, we find that $|K|$ is divisible by 7, so $m = |K/L| = 7^e$ for some e.

Now let $H = (G_1)'$, the derived group of G_1, which is the unique normal subgroup of index 6 in G_1 for which the corresponding quotient is cyclic of order 6. Then H contains both J and K, since both $G_1/K \cong A_3 \wr S_2$ and $G_1/J \cong C_7 \rtimes C_6$ have a cyclic quotient of order 6. Moreover, $H = JK$ since J is maximal in H and $K \not\subseteq J$, and so $J \cap K$ has index $|JK/K| = |H/K| = 3$ in J and index $|JK/J| = |H/J| = 7$ in K. It follows that $(J \cap K)/L$ has order $|K/L|/7 = m/7 = 7^{e-1}$, and index 3 in J/L, and is therefore a normal Sylow 7-subgroup of J/L.

In particular, $(J \cap K)/L$ is characteristic in J/L, and therefore normal in G_4^1/L. Factoring it out gives a quotient $G_4^1/(J \cap K) \cong (G_4^1/L)/((J \cap K)/L)$ of G_4^1 of order

$(8 \cdot 18m)/(m/7) = 1008$, which is then a 4-arc-transitive group of automorphisms of a cubic graph of order $1008/24 = 42$. (In fact this will be a 3-fold cover of the Heawood graph.) There is, however, only one arc-transitive cubic graph of order 42, namely the graph F042 listed in the appendix of [10], but that graph is 1-arc-regular. Hence this possibility can be eliminated.

Similarly, if L is G_1-invariant but not G_2^1-invariant and the cover is 5-arc-transitive, then by [12, Proposition 30] or [10, Proposition 3.4], the cover is a regular cover of the Biggs-Conway graph. In particular, its 1-arc-regular group of automorphisms must have $\mathrm{PSL}(2,7)$ as a composition factor, and hence is insoluble. But on the other hand, G_1/L is a normal extension of an abelian group by $A_3 \wr S_2$ and is therefore soluble, so this possibility can also be eliminated.

Thus all of the covers obtained above are 1-, 2- or 3-arc-regular.

Next, we determine isomorphisms between the covering graphs that arise from the G_1-invariant subgroups in Table 5.1. When the subgroup is G_3-invariant, the cover is 3-arc-regular, and is then unique up to isomorphism. When the subgroup is G_2^1-invariant but not G_3-invariant, the cover is 2-arc-regular of type 2^1, but isomorphic to the cover that arises from (exactly) one other subgroup in the same row of the table. Similarly, when the subgroup is G_1-invariant but not G_2^1-invariant, the cover is 1-arc-regular, and isomorphic to the cover that arises from (exactly) one other subgroup in the same row of the table.

This completes the determination of abelian regular covers of $K_{3,3}$ via its arc-regular group of automorphisms, $A_3 \wr S_2$. To complete the analysis, we consider what happens with the other minimal arc-transitive group of automorphisms, which is the 2-arc-regular group generated by the cosets Nh and Nap of N in the quotient G_3/N, and isomorphic to a semi-direct product $(A_3 \times A_3) \rtimes C_4$.

In this case, we work inside the quotient G_2^2/N', using the linear transformations of $K = N/N' \cong Z^4$ induced by h and ap, which are as follows:

$$h \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad ap \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The quotient $G_2^2/N \cong (A_3 \times A_3) \rtimes C_4$ has the following character table:

Element order	1	2	3	3	4	4
Class size	1	9	4	4	9	9
ψ_1	1	1	1	1	1	1
ψ_2	1	1	1	1	-1	-1
ψ_3	1	-1	1	1	ξ	$-\xi$
ψ_4	1	-1	1	1	$-\xi$	ξ
ψ_5	4	0	-2	1	0	0
ψ_6	4	0	1	-2	0	0

where ξ is a primitive 4th root of 1.

Since the trace of the matrix induced by h is -2, it is immediately obvious that the character of the given representation is ψ_5, which is irreducible over $\mathbb{Q}$. It follows that for every prime $k \notin \{2,3\}$, the group $K/K^{(k)} \cong (\mathbb{Z}_k)^4$ has no non-trivial proper G_2^2-invariant subgroup. The same holds also for $k = 2$, since the mod 2 reductions of each of the characters ψ_1 to ψ_4 are all trivial.

Hence for any prime $k \neq 3$, if L is a G_2^2-invariant subgroup of K with index $|K : L|$ a power of k, then every layer of L has rank 0 or 4, and so $|K : L| = d^4$ and $K/L \cong \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$ for some d. Every such subgroup L, however, is also preserved by conjugation by p (and by a) and hence is G_3-invariant, and so gives one of the 3-arc-regular covers found earlier.

For $k = 3$, the group $K/K^{(k)}$ has just two non-trivial proper G_2^2-invariant subgroup, namely the cyclic subgroup generated by the image of $z = w_1w_2w_3w_4$ (which is centralized by h and inverted under conjugation by ap), and the subgroup generated by any three of $x = w_1w_3^{-1}w_4$, $y = w_2w_3^{-1}$, $u = w_1w_4^{-1}$ and $v = w_2w_3w_4^{-1}$, with h conjugating x, y, u and v to y^{-1}, xy^{-1}, v^{-1} and uv^{-1}, and ap conjugating x, y, u and v to v^{-1}, u^{-1}, y and $x^{-1}y$, respectively.

Next, an easy exercise shows that $K/K^{(9)}$ has no G_2^2-invariant cyclic subgroup of order 9; equivalently, if the top layer of a G_2^2-invariant subgroup L is cyclic, then its next layer must have rank at least 3. Similarly, $K/K^{(9)}$ has no G_2^2-invariant rank 3 subgroup isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$; equivalently, if the top layer of a G_2^2-invariant subgroup L has rank 3, then its next layer must have (full) rank 4.

Hence the only G_2^2-invariant subgroups of K with 3-power index are among those given in rows 1, 4, 5 and 6 of Table 5.1. On the other hand, if L is one of the second or third kinds of subgroup in row 6 of Table 5.1, generated by $z^dw_4^{\pm 3d}$, y^{3d}, u^{3d} and w_4^{9d}, then L is not G_2^2-invariant, since conjugation by ap takes $z^dw_4^{\pm 3d}$ to $z^{-d}w_2^{\mp 3d}$, which does not lie in L. In particular, every G_2^2-invariant subgroup of K with 3-power index is also G_3-invariant, and again gives one of the 3-arc-regular covers found earlier.

In other words, every abelian regular cover of $K_{3,3}$ obtainable via the 2-arc-regular subgroup $(A_3 \times A_3) \rtimes C_4$ of $S_3 \wr S_2$ is 3-arc-regular, and hence is obtainable via the 1-arc-regular subgroup $A_3 \wr S_2$.

Thus we have the following theorem:

Theorem 5.1 *Let $m = k^e$ be any power of a prime k, with $e > 0$. Then the arc-transitive regular covers of the complete bipartite graph $K_{3,3}$ with abelian covering group of exponent m are as follows:*

(a) *If $k \equiv 2 \bmod 3$, there are exactly $e+1$ such covers, namely a 3-arc-regular cover with covering group $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, and one 2-arc-regular cover of type 2^1 with covering group $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ for each proper divisor ℓ of m.*

(b) *If $k \equiv 1 \bmod 3$, then there are exactly $\frac{3}{2}e(e+1)+1$ such covers, namely a 3-arc-regular cover with covering group $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, and one 2-arc-regular cover of type 2^1 with covering group $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ for each proper divisor ℓ of m, plus one 1-arc-regular cover with covering group $\mathbb{Z}_j \oplus \mathbb{Z}_j \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m$ for each divisor j of m and each proper divisor ℓ of m, plus one 1-arc-regular cover with covering group $\mathbb{Z}_j \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ for each pair $\{j, \ell\}$ of distinct proper divisors of m.*

(c) *If $k = 3$ and $e \geq 3$, then there are exactly $8e-5$ such covers, namely four 3-arc-regular covers with covering groups $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, $\mathbb{Z}_{m/3} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, $\mathbb{Z}_{m/3} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$ and $\mathbb{Z}_{m/9} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$, plus one 2-arc-regular cover of type 2^1 with covering group $\mathbb{Z}_{m/9} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$, plus one 2-arc-regular cover of type 2^1 with covering group $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ for each proper divisor ℓ of m, plus two 2-arc-regular covers of type 2^1 with covering groups $\mathbb{Z}_\ell \oplus \mathbb{Z}_{3\ell} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ and $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$ for each proper divisor ℓ of $m/3$, plus three pairwise non-isomorphic 2-arc-regular covers of type 2^1 with covering groups $\mathbb{Z}_\ell \oplus \mathbb{Z}_{3\ell} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$ for each proper divisor ℓ of $m/9$, plus two 1-arc-regular covers with covering groups $\mathbb{Z}_\ell \oplus \mathbb{Z}_{3\ell} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ and $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$ for each proper divisor ℓ of $m/3$.*

(c) *If $k = 3$ and $e = 2$, then there are exactly 11 such covers, namely four 3-arc-regular covers with covering groups $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$, plus five 2-arc-regular covers of type 2^1 with covering groups $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$, $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_9$, and two 1-arc-regular covers with covering groups $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_9$.*

(d) *If $k = 3$ and $e = 1$, then there are exactly 4 such covers, namely three 3-arc-regular covers with covering groups $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_3$, and one 2-arc-regular covers of type 2^1 with covering group $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.*

6 Arc-transitive abelian covers of Q_3

In this section, we classify all the arc-transitive abelian regular covering graphs of the 3-cube Q_3 (the 1-skeleton of a 3-dimensional cube). We know that Q_3 is 2-arc-transitive, and belongs to the class 2^1. Its automorphism group is the direct product

$S_4 \times C_2$, of order 48, and the only other arc-transitive groups of automorphisms of Q_3 are the subgroups S_4 and $A_4 \times C_2$, each of which acts regularly on the arcs of Q_3.

Take $G_2^1 = \langle\, h, a, p \mid h^3 = a^2 = p^2 = (hp)^2 = [a, p] = 1\,\rangle$. This group has two normal subgroups of index 48, both with quotient $S_4 \times C_2$, but these are interchanged by the outer automorphism of G_2^1 that takes the generators h, a and p to h, ap and p, and so without loss of generality we can take either one of them. We will take N to be the normal subgroup of index 48 that is freely generated by

$$w_1 = (ha)^4, \quad w_2 = (ah)^4, \quad w_3 = h^{-1}ahahahah^{-1},$$
$$w_4 = ah^{-1}ahahahah^{-1}a \quad \text{and} \quad w_5 = (hah^{-1}a)^3.$$

Easy calculations show that the generators h, a and p act by conjugation as follows:

$$\begin{array}{lcllcllcl}
h^{-1}w_1h & = & w_2 & a^{-1}w_1a & = & w_2 & p^{-1}w_1p & = & w_2^{-1} \\
h^{-1}w_2h & = & w_3 & a^{-1}w_2a & = & w_1 & p^{-1}w_2p & = & w_1^{-1} \\
h^{-1}w_3h & = & w_1 & a^{-1}w_3a & = & w_4 & p^{-1}w_3p & = & w_3^{-1} \\
h^{-1}w_4h & = & w_2^{-1}w_5^{-1}w_3^{-1} & a^{-1}w_4a & = & w_3 & p^{-1}w_4p & = & w_4^{-1} \\
h^{-1}w_5h & = & w_4w_5^{-1}w_3^{-1} & a^{-1}w_5a & = & w_5^{-1} & p^{-1}w_5p & = & w_3w_5w_4^{-1}.
\end{array}$$

Now take the quotient G_2^1/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^5$ by the group $G_2^1/N \cong S_4 \times C_2$. The two subgroups of the latter group that act regularly on the arcs of Q_3 are the quotients (mod N) of the subgroups $\langle h, a\rangle$ and $\langle h, ap\rangle$, isomorphic to S_4 and $A_4 \times C_2$, respectively. Also replace the generators h, a, p and all w_i by their images in G_2^1/N', and let K denote the subgroup N/N'.

By the above observations, the generators h, a and ap induce linear transformations of the free abelian group $K \cong \mathbb{Z}^5$ as follows:

$$h \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{and} \quad ap \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}.$$

Next, the character table of the group S_4 is given below:

Element order	1	2	2	3	4
Class size	1	3	6	8	6
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	2	0	-1	0
χ_4	3	-1	-1	0	1
χ_5	3	-1	1	0	-1

By inspecting traces of the matrices of orders 2, 2, 3 and 4 induced by each of $(ha)^2$, a, h and ha, we see that the character of the 5-dimensional representation of S_4 over $\mathbb{Q}$ associated with the above action of $G = \langle h, a\rangle$ on K is the sum of the characters χ_3 and χ_4, of irreducible representations over $\mathbb{Q}$ of dimensions 2 and 3 respectively.

It follows that for every prime $k \notin \{2, 3\}$, the group $K/K^{(k)} \cong (\mathbb{Z}_k)^5$ is the direct sum of two G-invariant subgroups of ranks 2 and 3, and these are its only non-trivial proper G-invariant subgroups.

In fact, the rank 2 subgroup of K generated by $u = w_1w_2^{-1}w_4w_5^{-1}$ and $v = w_3w_4$ is normal in $\langle h, a\rangle$, with $u^h = v^{-1}$, $v^h = uv^{-1}$, $u^a = u^{-1}v$ and $v^a = v$, while the rank 3 subgroup generated by $x = w_1w_2w_4$, $y = w_3w_4^{-1}$ and $z = w_5$ is normal in $\langle h, a\rangle$, with $x^h = z^{-1}$, $y^h = xyz$, $z^h = (yz)^{-1}$, $x^a = xy$, $y^a = y^{-1}$ and $z^a = z^{-1}$.

In the quotient $G/K^{(k)}$ for prime $k \notin \{2, 3\}$, the image of the rank 2 subgroup $\langle u, v\rangle$ has no non-trivial G-invariant cyclic subgroup, and hence is irreducible, and similarly, the image of the rank 3 subgroup $\langle x, y, z\rangle$ has no non-trivial proper G-invariant subgroup, and hence is irreducible.

For $k = 2$, the image of the rank 2 subgroup $\langle u, v\rangle$ in $G/K^{(k)}$ is irreducible as an G-invariant subgroup, but in this case, it is also contained in the image of the rank 3 subgroup $\langle x, y, z\rangle$ (since $K^{(2)}xz = K^{(2)}w_1w_2w_4w_5 = K^{(2)}u$ and $K^{(2)}y = K^{(2)}w_3w_4^{-1} = K^{(2)}v$), as well as in a third non-trivial proper G-invariant subgroup, namely the rank 4 subgroup generated by the images of the products w_iw_j for all i, j (which contains the image of $\langle u, v\rangle$ but not the image of $\langle x, y, z\rangle$).

In contrast, for $k = 3$ the image of the rank 3 subgroup $\langle x, y, z\rangle$ in $G/K^{(k)}$ is irreducible as an G-invariant subgroup, while the image of the rank 2 subgroup $\langle u, v\rangle$ contains a non-trivial cyclic G-invariant subgroup, namely the subgroup generated by the image of $uv = w_1w_2^{-1}w_3w_4^2w_5^{-1}$, since $(uv)^h = w_2w_3^{-1}w_1w_2^{-2}w_5^{-2}w_3^{-2}w_4^{-1}w_5w_3 = w_1w_2^{-1}w_3^{-2}w_4^{-1}w_5^{-1} = uv$ and $(uv)^a = w_2w_1^{-1}w_4w_3^2w_5 = w_1^{-1}w_2w_3^{-1}w_4w_5 = (uv)^{-1}$. Hence in this case, we have also a fourth non-trivial proper G-invariant subgroup, namely the rank 4 subgroup generated by the images of x, y, z and uv.

As before, analogous observations hold also for each other layer K_i/K_{i+1} of K.

So now suppose $m = k^e$ is any prime-power greater than 1, and suppose $L/K^{(m)}$ is any non-trivial normal subgroup of $G/K^{(m)}$ contained in $K/K^{(m)}$.

If $k \neq 2, 3$, then it is easy to see that there exist divisors c and d of $m = k^e$ such that L is generated by u^c, v^c, x^d, y^d and z^d, and $L/K^{(m)} \cong (\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^3$.

When $k = 3$, we note that conjugation by h and a behave well, in that they take the triple $(x^{k^t}, y^{k^t}, z^{k^t})$ to $(z^{-k^t}, (xyz)^{k^t}, (yz)^{-k^t}) = ((z^{k^t})^{-1}, x^{k^t}y^{k^t}z^{k^t}, (y^{k^t})^{-1}(z^{k^t})^{-1})$ and to $((xy)^{k^t}, y^{-k^t}, z^{-k^t}) = (x^{k^t}y^{k^t}, (y^{k^t})^{-1}, (z^{k^t})^{-1})$ respectively, for every positive integer t. The same kind of thing happens also with the pair (u, v). On the other hand, conjugation by h and a take uv to $uv^{-2} = (uv)v^{-3}$ and $u^{-1}v^2 = (uv)^{-1}v^3$, and hence if any layer L_i/L_{i+1} of a G-invariant subgroup L of $K/K^{(m)}$ has rank 1 or 4 (generated by images of $(uv)^i$ or of $(uv)^i, x^i, y^i$ and z^i) then its next layer must have rank 2 or 5 (generated by images of $(uv)^{3i}$ and v^{3i} or of $(uv)^{3i}, v^{3i}, x^{3i}, y^{3i}$ and z^{3i}).

The case $k = 2$ is not quite so straightforward. In this case, each layer can have rank 0, 1, 2, 3 or 4, depending on the layers above it. To see exactly what happens, it is helpful to consider the case $m = 8 = 2^3$. It is an easy exercise (using Magma if necessary) to show that $K/K^{(8)}$ contains exactly 35 subgroups that are normal in $G/K^{(8)}$, and these may be summarised as follows:

- the group $K/K^{(8)} \cong (\mathbb{Z}_8)^5$ itself, generated by all the $\overline{w_i}$,
- three other homocyclic subgroups $K^{(2^i)}/K^{(8)} \cong (\mathbb{Z}_{2^{3-i}})^5$, generated by all the $\overline{w_i}^{2^i}$,
- one subgroup isomorphic to $(\mathbb{Z}_8)^4 \oplus \mathbb{Z}_4$, generated by $\{\overline{u}, \overline{v}, \overline{w_1w_4}, \overline{w_1w_4}, \overline{w_5}^2\}$;
- one subgroup isomorphic to $(\mathbb{Z}_4)^4 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}^2, \overline{v}^2, (\overline{w_1w_4})^2, (\overline{w_2w_4})^2, \overline{w_5}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_2)^4$, generated by $\{\overline{u}^4, \overline{v}^4, (\overline{w_1w_4})^4, (\overline{w_2w_4})^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^3 \oplus (\mathbb{Z}_4)^2$, generated by $\{\overline{x}, \overline{y}, \overline{z}, \overline{w_2}^2, \overline{w_4}^2\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^3 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{x}, \overline{y}, \overline{z}, \overline{w_2}^4, \overline{w_4}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_4)^3 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{x}^2, \overline{y}^2, \overline{z}^2, \overline{w_2}^4, \overline{w_4}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_4)^3$, generated by $\{\overline{u}, \overline{v}, \overline{w_2}^2, \overline{w_4}^2, \overline{w_5}^2\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^3$, generated by $\{\overline{u}, \overline{v}, \overline{w_2}^4, \overline{w_4}^4, \overline{w_5}^4\}$ and $\{\overline{u}\,\overline{w_2}^2\overline{w_5}^2, \overline{v}\,\overline{w_4}^2\overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4, \overline{w_5}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_2)^3$, generated by $\{\overline{u}^2, \overline{v}^2, \overline{w_2}^4, \overline{w_4}^4, \overline{w_5}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_4)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}, \overline{v}, (\overline{w_1w_4})^2, (\overline{w_2w_4})^2, \overline{w_5}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{u}, \overline{v}, \overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4\}$ and $\{\overline{u}\,\overline{w_2}^2, \overline{v}\,\overline{w_4}^{-2}, \overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{u}, \overline{v}, (\overline{w_1w_4})^4, (\overline{w_2w_4})^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{u}^2, \overline{v}^2, (\overline{w_1w_4})^4, (\overline{w_2w_4})^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^3$, generated by $\{\overline{x}, \overline{y}, \overline{z}\}$, plus two other subgroups isomorphic to $(\mathbb{Z}_{2^{3-i}})^3$, generated by $\{\overline{x}^{2^i}, \overline{y}^{2^i}, \overline{z}^{2^i}\}$, for $i = 1, 2$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_4$, generated by $\{\overline{u}\,\overline{w_2}^2, \overline{v}\,\overline{w_4}^{-2}, \overline{w_5}^2\}$ and $\{\overline{u}\,\overline{w_2}^{-2}, \overline{v}\,\overline{w_4}^2, \overline{w_5}^2\}$;
- two subgroups isomorphic to $(\mathbb{Z}_4)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}^2, \overline{v}^2, \overline{w_5}^4\}$ and $\{\overline{u}^2\,\overline{w_2}^4, \overline{v}^2\,\overline{w_4}^4, \overline{w_5}^4\}$;
- four subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}, \overline{v}, \overline{w_5}^4\}$, $\{\overline{u}\,\overline{w_2}^4, \overline{v}\,\overline{w_4}^4, \overline{w_5}^4\}$, $\{\overline{u}\,\overline{w_2}^2\overline{w_5}^2, \overline{v}^2\,\overline{w_4}^{-2}\overline{w_5}^2, \overline{w_5}^4\}$ and $\{\overline{u}\,\overline{w_2}^{-2}\overline{w_5}^2, \overline{v}^2\,\overline{w_4}^2\overline{w_5}^2, \overline{w_5}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2$, generated by $\{\overline{u}, \overline{v}\}$ and $\{\overline{u}\,\overline{w_2}^4, \overline{v}\,\overline{w_4}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_4)^2$, generated by $\{\overline{u}^2, \overline{v}^2\}$ and $\{\overline{u}^2\,\overline{w_2}^4, \overline{v}^2\,\overline{w_4}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_2)^2$, generated by $\{\overline{u}^4, \overline{v}^4\}$.

If we let T_0, T_1, T_2, T_3 and T_4 denote the five possibilities for a G-invariant subgroup of $K/K^{(2)}$ (of ranks 0, 2, 3, 4 and 5 respectively), then once again we can represent each of the 2-layer combinations as a pair (T_i, T_j). The pairs that arise in this case can be found from looking at G-invariant subgroups of $K/K^{(4)}$ (or equivalently, the top two layers or the bottom two layers of $K/K^{(8)}$), and these are easily found to be the following, counted according to their frequency: (T_4, T_4) once; (T_3, T_4) once; (T_2, T_2) once; (T_2, T_4) once; (T_1, T_1) twice; (T_1, T_2) twice; (T_1, T_3) once; (T_1, T_4) once; and (T_0, T_0), (T_0, T_1), (T_0, T_2), (T_0, T_3) and (T_0, T_4), once each.

The 3-layer combinations that arise from the above list of G-invariant subgroups of $K/K^{(8)}$ can now be summarised with their associated frequencies, as follows:

- (T_4, T_4, T_4), once, • (T_0, T_4, T_4), (T_0, T_0, T_4) and (T_0, T_0, T_0), once each,
- (T_3, T_4, T_4), once, • (T_3, T_3, T_4), once, • (T_0, T_0, T_3), once,
- (T_2, T_4, T_4), once, • (T_2, T_2, T_4), once, • (T_0, T_2, T_4), once,
- (T_1, T_4, T_4), once, • (T_1, T_1, T_4), twice, • (T_0, T_1, T_4), once,
- (T_1, T_3, T_4), once, • (T_1, T_2, T_4), twice, • (T_1, T_1, T_3), once,
- (T_0, T_1, T_3), once, • (T_2, T_2, T_2), (T_0, T_2, T_2) and (T_0, T_0, T_2), once each,
- (T_1, T_2, T_2), twice, • (T_0, T_1, T_2), twice, • (T_1, T_1, T_2), four times,
- (T_1, T_1, T_1), twice, • (T_0, T_1, T_1), twice, • (T_0, T_0, T_1), once.

The same argument shows that each 'triple-layer' section K_i/K_{i+3} of K has exactly 35 G-invariant subgroups, analogous to those in the summary list above, when $k = 2$.

We can now put these observations together to find all possibilities for a normal subgroup L of $G = \langle h, a \rangle$ contained in K with index $|K\!:\!L|$ being a power $m = k^e$ of a prime k.

When $k = 2$, the layers of any such L must consist of (say) e_0 copies of T_0, followed by e_1 copies of T_1, followed by e_2 copies of T_2, followed by e_3 copies of T_3, followed by e_4 copies of T_4, with $e_0 + e_1 + e_2 + e_3 + e_4 = e$, and $e_i \geq 0$ for all i, but $e_3 \leq 1$, and $e_2 e_3 = 0$. Moreover, any such combination determines a unique L, except in some cases where a pair of successive layers is of the form (T_1, T_1) or (T_1, T_2) and no layer is T_3, namely as follows:

a) If $(e_1, e_2, e_3) = (2, 1, 0)$, so that L is a tower of two copies of T_1 on top of a single copy of T_2, on top of a tower of any number of copies of T_4, then there are four possibilities for L,

b) If $e_3 = 0$ and either $e_1 > 1$ or $e_1 e_2 > 0$, but $(e_1, e_2) \neq (2, 1)$, then there are two possibilities for L.

All the possibilities for L are listed in Table 6.1.

Index $\lvert K:L\rvert$	Generating set for L	Quotient K/L
$d^5 = k^{5t}$, for any k	$\{w_1^d,\ w_2^d,\ w_3^d,\ w_4^d,\ w_5^d\}$	$(\mathbb{Z}_d)^5$
$c^2d^3 = k^{2s}k^{3t}$, with $s \neq t$, for any $k > 2$	$\{u^c,\ v^c,\ x^d,\ y^d,\ z^d\}$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^3$
$8d^3 = 2^{5t+3}$	$\{u^d,\ v^d,\ x^{2d},\ y^{2d},\ z^{2d}\}$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{2d})^3$
$c^2d^3 = 2^{2s+3t}$, with $s+1 < t$	$\{u^c,\ v^c,\ w_2^d,\ w_4^d,\ w_5^d\}$ or $\{u^c w_2^{\frac{d}{2}} w_5^{\frac{d}{2}},\ v^c w_4^{\frac{d}{2}} w_5^{\frac{d}{2}},\ w_2^d,\ w_4^d,\ w_5^d\}$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^3$
$c^3d^2 = 2^{3s+2t}$, with $s < t$	$\{x^c,\ y^c,\ z^c,\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_d)^2$
$4c^2d^3 = 2^{2s}2^{3t+2}$, with $s < t$	$\{u^c,\ v^c,\ w_5^d,\ w_2^{2d},\ w_4^{2d}\}$ or $\{u^c w_2^d,\ v^c w_4^d,\ w_5^d,\ w_2^{2d},\ w_4^{2d}\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{2d})^2$
$2c^3d^2 = 2^{3s+1}2^{2t}$, with $s+2 < t$	$\{u^c w_2^{2c},\ v^c w_4^{-2c},\ w_5^{2c},\ w_2^d,\ w_4^d\}$ or $\{u^c w_2^{2c-\frac{d}{2}},\ v^c w_4^{-2c+\frac{d}{2}},\ w_5^{2c},\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_{2c} \oplus (\mathbb{Z}_d)^2$
$4c^3d^2 = 2^{3s+2}2^{2t}$, with $s+2 < t$	$\{u^c w_2^{2c} w_5^{2c},\ v^c w_4^{-2c} w_5^{2c},\ w_5^{4c},\ w_2^d,\ w_4^d\}$ or $\{u^c w_2^{2c-\frac{d}{2}} w_5^{2c},\ v^c w_4^{-2c+\frac{d}{2}} w_5^{2c},\ w_5^{4c},\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_{4c} \oplus (\mathbb{Z}_d)^2$
$4c^2d^3 = 2^{2s}2^{3t+1}$, with $s < t$	$\{u^c,\ v^c,\ (w_1w_4)^d,\ (w_2w_4)^d,\ w_5^{2d}\}$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^2 \oplus \mathbb{Z}_{2d}$
$3c^3d^2 = 3^{3s}3^{2t+1}$, any s and t	$\{x^c,\ y^c,\ z^c,\ (uv)^d,\ v^{3d}\}$	$(\mathbb{Z}_c)^3 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{3d}$

Table 6.1: Possibilities for $\langle h,a\rangle$-invariant subgroup L of K when $\langle h,a\rangle/K \cong S_4$

Next, we consider which of the $\langle h,a\rangle$-invariant subgroups of $K/K^{(m)}$ are normalized by the additional generator p of the larger group G_2^1. Here we will work inside the group G_2^1/N', and adopt the same notation for images of elements in this group.

Note that conjugation by p (which lies in G_2^1/N' but not G_1/N') takes

$$\begin{aligned}
u = w_1w_2^{-1}w_4w_5^{-1} &\mapsto w_2^{-1}w_1w_4^{-1}w_3^{-1}w_5^{-1}w_4 = u^{-1},\\
v = w_3w_4 &\mapsto w_3^{-1}w_4^{-1} = v^{-1},\\
x = w_1w_2w_4 &\mapsto w_2^{-1}w_1^{-1}w_4^{-1} = x^{-1},\\
y = w_3w_4^{-1} &\mapsto w_3^{-1}w_4 = y^{-1},\\
z = w_5 &\mapsto w_3w_5w_4^{-1} = yz,
\end{aligned}$$

and also preserves the subgroup generated by the products w_iw_j for all i,j. Moreover, when $k=3$, conjugation by p takes $uv = w_1w_2^{-1}w_3w_4^2w_5^{-1}$ to $w_1w_2^{-1}w_3^{-2}w_4^{-1}w_5^{-1} = uv$.

Hence all of the $\langle h,a\rangle$-invariant subgroups of $K/K^{(k)}$ that we met above for each prime k are also $\langle h,a,p\rangle$-invariant. In fact it is not difficult to see that all of the $\langle h,a\rangle$-invariant subgroups summarised in Table 6.1 are G_2^1-invariant.

It follows that all of the resulting covers of Q_3 admit a 2-arc-regular group of automorphisms. In particular, none of these covers can be 4- or 5-arc-transitive, by [12, Theorem 3]. Also by the same arguments as used for K_4 (in Section 4), none of them can be 3-arc-transitive, because the subgroup N itself is not normal in G_3. Thus all of the covers resulting from the above possibilities for the subgroup L are 2-arc-regular. Moreover, again since the subgroup N itself is not normal in G_3, each of them is unique up to isomorphism.

This completes the determination of abelian regular covers of Q_3 obtainable via the arc-regular group of automorphisms isomorphic to S_4.

Next, we consider what happens with the other minimal arc-transitive group of automorphisms, namely the arc-regular group generated by the cosets Nh and Nap of N in the quotient G_2^1/N, which is isomorphic to the direct product $A_4 \times C_2$.

The character table of $A_4 \times C_2$ is as follows:

Element order	1	2	2	2	3	3	6	6
Class size	1	1	3	3	4	4	4	4
ψ_1	1	1	1	1	1	1	1	1
ψ_2	1	-1	1	-1	1	1	-1	-1
ψ_3	1	1	1	1	λ	λ^2	λ	λ^2
ψ_4	1	1	1	1	λ^2	λ	λ^2	λ
ψ_5	1	-1	1	-1	λ	λ^2	$-\lambda$	$-\lambda^2$
ψ_6	1	-1	1	-1	λ^2	λ	$-\lambda^2$	$-\lambda$
ψ_7	3	3	-1	-1	0	0	0	0
ψ_8	3	-3	-1	1	0	0	0	0

where λ is a primitive cube root of 1.

By inspecting traces, we see that the character of the 5-dimensional representation of $A_4 \times C_2$ over $\mathbb{Q}$ associated with the above action of $\langle h, ap\rangle$ on K is the character $\psi_5 + \psi_6 + \psi_7$, which is reducible to the sum of $\psi_5 + \psi_6$ and ψ_7, again the characters of irreducible representations of dimensions 2 and 3 over $\mathbb{Q}$.

For every prime $k \notin \{2, 3\}$, the group $K/K^{(k)} \cong (\mathbb{Z}_k)^5$ is a direct sum of the subgroups of ranks 2 and 3 generated by the images of u and v on one hand, and of x, y and z on the other. Both of these subgroups are $\langle h, ap\rangle$-invariant, since $u^{ap} = u^{-1}$, $v^{ap} = v^{-1}$, $x^{ap} = x^{-1}y^{-1}$, $y^{ap} = y$ and $z^{ap} = y^{-1}z^{-1}$. Moreover, these two $\langle h, ap\rangle$-invariant subgroups are irreducible, unless $\mathbb{Z}_k$ contains a cube root of 1, in which case the rank 2 subgroup is reducible.

When $k > 2$ and $k \equiv 2 \bmod 3$, it follows that the $\langle h, ap\rangle$-invariant subgroups with index in K being a power of k are precisely the same as the $\langle h, a\rangle$-invariant subgroups, given in the first two rows of Table 6.1.

When $k > 2$ and $k \equiv 1 \bmod 3$ (and $m = k^e$ for some $e > 0$), the rank 2 subgroup of the quotient $K/K^{(m)}$ generated by the images of u and v is the direct sum of two

$\langle h, ap\rangle$-invariant rank 1 subgroups, generated by the images of v_λ and v_{λ^2}, where λ is a primitive cube root of 1 in $\mathbb{Z}_m$ and $v_t = w_1 w_2^{-1} w_3^t w_4^{1+t} w_5^{-1}$ for $t \in \{\lambda, \lambda^2\}$. Note that since $t^2+t+1 = 0$ in $\mathbb{Z}_m$, we have $v_t^h = w_2 w_3^{-1} w_1^t w_2^{-1-t} w_5^{-1-t} w_3^{-1-t} w_4^{-1} w_5 w_3 = w_1^t w_2^{-t} w_3^{t^2} w_4^{-1} w_5^{-t} = v_t^t$ and $v_t^{ap} = w_1^{-1} w_2 w_4^{-t} w_3^{-1-t} w_4^{-1} w_5 w_3 = w_1^{-1} w_2 w_3^{-t} w_4^{-1-t} w_5 = v_t^{-1}$ modulo $K^{(k)}$. On the other hand, the rank 3 subgroup of $K/K^{(m)}$ generated by the images of x, y and z remains irreducible under conjugation by $\langle h, ap\rangle$.

When $k = 3$, the $\langle h, ap\rangle$-invariant subgroup generated by the images of x, y and z remains irreducible, and also the rank 1 subgroup generated by the image of uv is $\langle h, ap\rangle$-invariant, with $(uv)^h = uv$ and $(uv)^{ap} = w_1^{-1} w_2 w_4^{-1} w_3^{-2} w_4^{-1} w_5 w_3 = (uv)^{-1}$. Hence the $\langle h, ap\rangle$-invariant subgroups with index in K being a power of 3 are precisely the same as the $\langle h, a\rangle$-invariant subgroups, given in the first two rows and the last row of Table 6.1.

When $k = 2$ on the other hand, the $\langle h, ap\rangle$-invariant rank 2 subgroup of $K/K^{(k)}$ generated by the images of u and v remains irreducible (since $u^h = v^{-1}$ and $v^h = uv^{-1}$ while $u^{ap} = u^{-1}$ and $v^{ap} = v^{-1}$). In fact, the $\langle h, ap\rangle$-invariant subgroups of $K/K^{(2)}$ are precisely the same as the $\langle h, a\rangle$-invariant subgroups, namely the ones we called T_0, T_1, T_2, T_3 and T_4 above. Other possibilities arise, however, for $\langle h, ap\rangle$-invariant subgroups of $K/K^{(m)}$ when m is a larger power of 2.

Again it is helpful to consider the case $m = 8 = 2^3$, to see what happens. It is an easy exercise (using Magma if necessary) to show that $K/K^{(8)}$ contains exactly 16 subgroups that are preserved under conjugation by h and ap but not by a (or p), and these may be summarised as follows:

- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^3$, generated by $\{\overline{u}\,\overline{w_2}^2\overline{w_4}^2, \overline{v}\,\overline{w_2}^2\overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4, \overline{w_5}^4\}$ and $\{\overline{u}\,\overline{w_4}^2\overline{w_5}^2, \overline{v}\,\overline{w_2}^2\overline{w_4}^2, \overline{w_2}^4, \overline{w_4}^4, \overline{w_5}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{u}\,\overline{w_4}^2, \overline{v}\,\overline{w_2}^2\overline{w_4}^2, \overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4\}$ and $\{\overline{u}\,\overline{w_2}^2\overline{w_4}^2, \overline{v}\,\overline{w_2}^2, \overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_4$, generated by $\{\overline{u}\,\overline{w_2}^2\overline{w_4}^4, \overline{v}\,\overline{w_2}^4\overline{w_4}^2, \overline{w_5}^2\}$ and $\{\overline{u}\,\overline{w_2}^6\overline{w_4}^4, \overline{v}\,\overline{w_2}^4\overline{w_4}^{-2}, \overline{w_5}^2\}$;
- two subgroups isomorphic to $(\mathbb{Z}_4)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}^2\,\overline{w_4}^4, \overline{v}^2\,\overline{w_2}^4\overline{w_4}^4, \overline{w_5}^4\}$ and $\{\overline{u}^2\,\overline{w_2}^4\overline{w_4}^4, \overline{v}^2\,\overline{w_2}^4, \overline{w_5}^4\}$;
- four subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}\,\overline{w_4}^4, \overline{v}\,\overline{w_2}^4\overline{w_4}^4, \overline{w_5}^4\}$, $\{\overline{u}\,\overline{w_2}^4\overline{w_4}^4, \overline{v}\,\overline{w_2}^4, \overline{w_5}^4\}$, $\{\overline{u}\,\overline{w_1}^{-2}\overline{w_4}^2, \overline{v}\,\overline{w_1}^6\overline{w_2}^2, \overline{w_5}^4\}$ and $\{\overline{u}\,\overline{w_1}^2\overline{w_4}^{-2}, \overline{v}\,\overline{w_1}^2\overline{w_2}^6, \overline{w_5}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2$, generated by $\{\overline{u}\,\overline{w_2}^4\overline{w_4}^4, \overline{v}\,\overline{w_2}^4\overline{w_5}^4\}$ and $\{\overline{u}\,\overline{w_4}^4\overline{w_5}^4, \overline{v}\,\overline{w_2}^4\overline{w_4}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_4)^2$, generated by $\{\overline{u}^2\,\overline{w_2}^4\overline{w_4}^4, \overline{v}^2\,\overline{w_2}^4\overline{w_5}^4\}$ and $\{\overline{u}^2\,\overline{w_4}^4\overline{w_5}^4, \overline{v}^2\,\overline{w_2}^4\overline{w_4}^4\}$.

The triples representing these 16 subgroups are: (T_1, T_1, T_4) twice, (T_1, T_2, T_4) twice, (T_1, T_2, T_2), twice, (T_0, T_1, T_2) twice, (T_1, T_1, T_2) four times, (T_1, T_1, T_1) twice, and (T_0, T_1, T_1) twice, respectively.

Using these observations (and again some more for the case $m = 16$ for clarity), we find that the only possibilities for a subgroup L of prime-power index in K that is $\langle h, ap\rangle$-invariant but not $\langle h, a\rangle$-invariant are those in Table 6.2.

Index $\lvert K:L\rvert$	Generating set for L	Quotient K/L
$b^3cd = k^{3r+s+t}$, with $s < t$, for $k \equiv 1 \bmod 3$	$\{x^b,\ y^b,\ z^b,\ (v_\lambda)^c,\ (v_{\lambda^2})^d\}$ or $\{x^b,\ y^b,\ z^b,\ (v_{\lambda^2})^c,\ (v_\lambda)^d\}$	$(\mathbb{Z}_b)^3 \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d$
$c^2d^3 = 2^{2s+3t}$, with $s+1 < t$	$\{u^c w_2^{\frac{d}{2}} w_4^{\frac{d}{2}},\ v^c w_2^{\frac{d}{2}} w_5^{\frac{d}{2}},\ w_2^d,\ w_4^d,\ w_5^d\}$ or $\{u^c w_4^{\frac{d}{2}} w_5^{\frac{d}{2}},\ v^c w_2^{\frac{d}{2}} w_4^{\frac{d}{2}},\ w_2^d,\ w_4^d,\ w_5^d\}$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^3$
$4c^2d^3 = 2^{2s}2^{3t+2}$, with $s+1 < t$	$\{u^c w_4^d,\ v^c w_2^d w_4^d,\ w_5^d,\ w_2^{2d},\ w_4^{2d}\}$ or $\{u^c w_2^d w_4^d,\ v^c w_2^d,\ w_5^d,\ w_2^{2d},\ w_4^{2d}\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{2d})^2$
$2c^3d^2 = 2^{3s+1}2^{2t}$, with $s+2 < t$	$\{u^c w_2^{2c} w_4^{\frac{d}{2}},\ v^c w_2^{\frac{d}{2}} w_4^{-2c+\frac{d}{2}},\ w_5^{2c},\ w_2^d,\ w_4^d\}$ or $\{u^c w_2^{2c+\frac{d}{2}} w_4^{\frac{d}{2}},\ v^c w_2^{\frac{d}{2}} w_4^{-2c},\ w_5^{2c},\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_{2c} \oplus (\mathbb{Z}_d)^2$
$4c^3d^2 = 2^{3s+2}2^{2t}$, with $s+3 < t$	$\{u^c w_1^{-2c} w_4^{-2c+\frac{d}{2}},\ v^c w_1^{2c+\frac{d}{2}} w_2^{2c},\ w_5^{4c},\ w_2^d,\ w_4^d\}$ or $\{u^c w_1^{-2c+\frac{d}{2}} w_4^{-2c},\ v^c w_1^{2c} w_2^{2c+\frac{d}{2}},\ w_5^{4c},\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_{4c} \oplus (\mathbb{Z}_d)^2$

Table 6.2: Additional possibilities for $\langle h, ap\rangle$-invariant subgroup L of K when $\langle h, a\rangle/K \cong S_4$ and $\langle h, ap\rangle/K \cong A_4 \times C_2$

The subgroups in the last four rows of Table 6.2 all come from the case $k = 2$. The 'layer combinations' for a subgroup L from each of these rows are respectively as follows:

- Row 2: a tower of two or more copies of T_1 on top of a tower of copies of T_4;
- Row 3: a tower of two or more copies of T_1 on top of a single copy of T_2 on top of a tower of copies of T_4;
- Row 4: a single copy of T_1 on top of a tower of two or more copies of T_2 on top of a tower of copies of T_4;
- Row 5: a tower of two copies of T_1 on top of a single copy of T_2 on top of a tower of copies of T_4.

Note that there are two possibilities in each case; the triple (T_1, T_1, T_2) which appeared four times in the summary for $m = 8$ occurs twice in each of rows 3 and 5.

The covering graphs of Q_3 corresponding to the subgroups of K in Table 6.2 admit a 1-arc-regular but not a 2-arc-regular group of automorphisms (since the subgroup is $\langle h, ap\rangle$-invariant but not $\langle h, a\rangle$-invariant). Hence in particular, by [12, Proposition 26] or [10, Proposition 2.3], none of them can be 3-arc-regular.

Next, suppose the subgroup L is G_1-invariant but not G_2^1-invariant, and the cover is 4-arc-transitive. In that case, by the same arguments as for $K_{3,3}$, the cover must be a regular cover of the Heawood graph, and since $G_1/K \cong A_4 \times C_2$ has a cyclic quotient of order 6, this implies the existence of a group of order $336 \cdot 4 = 1344$ acting 4-arc-transitively on a symmetric cubic graph of order $1344/24 = 56$. There are, however, only three symmetric cubic graphs of that order (namely the graphs F056A, F056B and F056C listed in [10]), and none of them is 4-arc-transitive. Hence this possibility can be eliminated.

Similarly, if L is G_1-invariant but not G_2^1-invariant and the cover is 5-arc-transitive, then by the same arguments as for $K_{3,3}$, the covering graph must be a regular cover of the Biggs-Conway graph, and hence the 1-arc-regular group $\langle h, ap\rangle/L$ of automorphisms must be insoluble, which is not the case.

Hence all of the resulting covers of Q_3 are 1-arc-regular.

Moreover, since conjugation by a (or p) interchanges the subgroups in each row of Table 6.2 in pairs, these covering graphs are isomorphic in pairs.

Thus we have the following theorem:

Theorem 6.1 *Let $m = k^e$ be any power of a prime k, with $e > 0$. Then the arc-transitive regular covers of the 3-cube Q_3 with abelian covering group of exponent m are as follows:*

(a) *If $k \equiv 2 \bmod 3$ and $k > 2$, there are exactly $2e + 1$ such covers, namely a 2-arc-regular cover with covering group $(\mathbb{Z}_m)^5$, plus one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_m)^3$ and one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^2$, for each proper divisor ℓ of m.*

(b) *If $k \equiv 1 \bmod 3$, then there are exactly $\frac{1}{2}e(e+1)+e^2+2e+1$ such covers, namely a 2-arc-regular cover with covering group $(\mathbb{Z}_m)^5$, plus one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_m)^3$ and one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^2$, for each proper divisor ℓ of m, plus one 1-arc-regular cover with covering group $\mathbb{Z}_j \oplus \mathbb{Z}_\ell \oplus (\mathbb{Z}_m)^3$ for each pair $\{j, \ell\}$ of distinct divisors of m, and one 1-arc-regular cover with covering group $(\mathbb{Z}_j)^3 \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m$ for each ordered pair (j, ℓ) of proper divisors of m.*

(c) *If $k = 3$, then there are exactly $4e+1$ such covers, namely a 2-arc-regular cover with covering group $(\mathbb{Z}_m)^5$, plus one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_m)^3$, one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^2$, one 2-arc-regular cover with covering group $\mathbb{Z}_\ell \oplus \mathbb{Z}_{3\ell} \oplus (\mathbb{Z}_m)^3$ and one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^3 \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$, for each proper divisor ℓ of m.*

(d) *If $k = 2$ and $e > 2$, then there are exactly $14e - 16$ such covers, namely a 2-arc-regular cover with covering group $(\mathbb{Z}_m)^5$, plus one 2-arc-regular cover with covering group $(\mathbb{Z}_{m/2})^2 \oplus (\mathbb{Z}_m)^3$, plus two 2-arc-regular covers and one 1-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_m)^3$, for each proper divisor ℓ of $m/2$, plus one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^2$ for each proper divisor ℓ of m, plus two 2-arc-regular covers and one 1-arc-regular cover with covering group $(\mathbb{Z}_{m/4})^2 \oplus \mathbb{Z}_{m/2} \oplus (\mathbb{Z}_m)^2$, plus four 2-arc-regular covers and two 1-arc-regular covers with covering group $(\mathbb{Z}_{m/8})^2 \oplus \mathbb{Z}_{m/2} \oplus (\mathbb{Z}_m)^2$, plus two 2-arc-regular covers and one 1-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus \mathbb{Z}_{m/2} \oplus (\mathbb{Z}_m)^2$ for each proper divisor ℓ of $m/8$, plus two 2-arc-regular covers and one 1-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus \mathbb{Z}_{2\ell} \oplus (\mathbb{Z}_m)^2$ for each* *proper divisor* ℓ *of* $m/4$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^2 \oplus \mathbb{Z}_{4\ell} \oplus (\mathbb{Z}_m)^2$ *for each proper divisor* ℓ *of* $m/8$*, and one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_{m/2})^2 \oplus \mathbb{Z}_m$ *for each proper divisor* ℓ *of* m*.*

(e) *If* $k = 2$ *and* $e = 2$*, then there are exactly* 12 *such covers, namely one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_2)^r \oplus (\mathbb{Z}_4)^{5-r}$ *for each* $r \in \{0, 2, 3, 4\}$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_4)^3$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2$*, plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$*, and one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_4)^2$*.*

(f) *If* $k = 2$ *and* $e = 1$*, then there are exactly* 4 *such covers, namely namely one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_2)^r$ *for each* $r \in \{2, 3, 4, 5\}$*.*

7 Arc-transitive abelian covers of the Petersen graph

In this section, we classify all the arc-transitive abelian regular covering graphs of the Petersen graph $\mathbb{P}$. Like $K_{3,3}$, this is a 3-arc-regular graph. Its automorphism group is the symmetric group S_5, of order 120, and the only other arc-transitive group of automorphisms is the subgroup A_5, which acts regularly on the 2-arcs, with type 2^1.

Take $G_3 = \langle\, h, a, p, q \mid h^3 = a^2 = p^2 = q^2 = [p, q] = [h, p] = (hq)^2 = apaq = 1 \,\rangle$, let $G = G_2^1 = \langle h, a, pq \rangle$, and let N be the unique normal subgroup of index 120 in G_3 (and index 60 in G_2^1), with quotients $G_3/N \cong S_5$ and $G_2^1/N \cong A_5$. Using Reidemeister-Schreier theory or the `Rewrite` command in Magma, we find that the subgroup N is free of rank 6, on generators

$$w_1 = (ha)^5, \quad w_2 = (ah)^5, \quad w_3 = h^{-1}ahahahahah^{-1}, \quad w_4 = pqahah^{-1}ahah^{-1}ah,$$
$$w_5 = pqhahah^{-1}ahah^{-1}ah^{-1} \quad \text{and} \quad w_6 = pqh^{-1}ahah^{-1}ahah^{-1}a.$$

Easy calculations show that the generators h, a, p and q act by conjugation as follows:

$$\begin{array}{lll}
h^{-1}w_1h = w_2 & a^{-1}w_1a = w_2 & p^{-1}w_1p = w_4^{-1} & q^{-1}w_1q = w_6 \\
h^{-1}w_2h = w_3 & a^{-1}w_2a = w_1 & p^{-1}w_2p = w_6^{-1} & q^{-1}w_2q = w_4 \\
h^{-1}w_3h = w_1 & a^{-1}w_3a = w_4w_3w_6 & p^{-1}w_3p = w_5^{-1} & q^{-1}w_3q = w_5 \\
h^{-1}w_4h = w_5 & a^{-1}w_4a = w_4^{-1} & p^{-1}w_4p = w_1^{-1} & q^{-1}w_4q = w_2 \\
h^{-1}w_5h = w_6 & a^{-1}w_5a = w_1^{-1}w_5^{-1}w_2^{-1} & p^{-1}w_5p = w_3^{-1} & q^{-1}w_5q = w_3 \\
h^{-1}w_6h = w_4 & a^{-1}w_6a = w_6^{-1} & p^{-1}w_6p = w_2^{-1} & q^{-1}w_6q = w_1.
\end{array}$$

Now take the quotient G_3/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^6$ by the group $G_3/N \cong S_5$, and replace the generators h, a, p, q and all w_i by their images in G_3/N'. Also let K denote the subgroup N/N', and let $G = G_2^1/N'$, so that G is an extension of K by A_5.

By the above observations, the generators h, a, p and q induce linear transformations of the free abelian group $K \cong \mathbb{Z}^6$ as follows:

$$h \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$p \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Next, the character table of the group A_5 is as follows:

Element order	1	2	3	5	5
Class size	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3	-1	0	β	α
χ_4	4	0	1	-1	1
χ_5	5	1	-1	0	0

where α and β are the zeroes of the polynomial $t^2 - t - 1$ (or in other words, the golden ratio $\frac{1+\sqrt{5}}{2}$ and its conjugate $\frac{1-\sqrt{5}}{2}$).

By inspecting traces of the matrices induced by each of a and h, we see that the character of the 6-dimensional representation of A_5 over $\mathbb{Q}$ associated with the above action of $\langle h, a, pq\rangle$ on K is the rational character $\chi_2 + \chi_3$, which is reducible over fields containing zeroes of the polynomial $t^2 - t - 1$.

So now let k be any odd prime. Then $\alpha^2 - \alpha - 1 = 0$ for some $\alpha \in \mathbb{Z}_k$ if and only if $(2\alpha - 1)^2 = 4\alpha^2 - 4\alpha + 1 = 5$ for some $\alpha \in \mathbb{Z}_k$, or equivalently, if and only if 5 is a quadratic residue mod k.

Hence if $k \equiv \pm 1 \bmod 5$, then the group $K/K^{(k)} \cong (\mathbb{Z}_k)^6$ is direct sum of two G-invariant subgroups of rank 3. In fact, these rank 3 subgroups are the images in $K/K^{(k)}$ of the two subgroups generated by $\{x_\alpha, y_\alpha, z_\alpha\}$ and $\{x_\beta, y_\beta, z_\beta\}$ where

$$x_\lambda = w_1 w_4 w_5 w_6^{-\lambda}, \quad y_\lambda = w_2 w_4^{-\lambda} w_5 w_6 \quad \text{and} \quad z_\lambda = w_3 w_4 w_5^{-\lambda} w_6$$

for $\lambda \in \{\alpha, \beta\}$, the set of zeroes of $t^2 - t - 1$ in $\mathbb{Z}_k$. Note that conjugation by h cyclically permutes x_λ, y_λ and z_λ, and conjugation by a inverts each of x_λ and y_λ and takes z_λ to $x_\lambda{}^\lambda y_\lambda{}^\lambda z_\lambda$, while conjugation by pq interchanges x_λ with y_λ^{-1} and inverts z_λ, for each $\lambda \in \{\alpha, \beta\}$. Moreover, these are the only non-trivial proper G-invariant subgroups of $K/K^{(k)}$.

If $k \equiv \pm 2 \bmod 5$ (and k is odd), then no such zeroes of $t^2 - t - 1$ exist in $\mathbb{Z}_k$, and the corresponding 6-dimensional representation of A_5 is irreducible over $\mathbb{Z}_k$, and it follows that $K/K^{(k)}$ has no non-trivial proper G-invariant subgroups. Note that this holds just as well when $k = 3$, since the representations χ_2 and χ_3 are distinct when defined over GF(9).

When $k = 5$, the mod k reductions of the characters χ_2 and χ_3 coincide, and we have just one non-trivial proper G-invariant subgroup of $K/K^{(k)}$, namely the rank 3 subgroup generated by the images of $x_\lambda = w_1w_4w_5w_6^{-\lambda}$, $y_\lambda = w_2w_4^{-\lambda}w_5w_6$ and $z_\lambda = w_3w_4w_5^{-\lambda}w_6$, where $\lambda = 2$ (the unique zero of $t^2 - t - 1$ in $\mathbb{Z}_5$).

For $k = 2$, the group $K/K^{(k)}$ has four non-trivial proper G-invariant subgroups, namely one of rank 4 generated by the images of w_1w_3, w_2w_3, w_4w_5 and w_5w_6, plus three of rank 5 containing the latter, with additional generators w_1, w_4 and w_1w_4, respectively. (This is left as an exercise, easily verifiable with the help of Magma.)

As before, analogous observations hold also for each other layer K_i/K_{i+1} of K.

Next, suppose $m = k^e$ is any prime-power greater than 1, and suppose $L/K^{(m)}$ is any non-trivial normal subgroup of $G/K^{(m)}$ contained in $K/K^{(m)}$.

If k is odd and $k \equiv \pm 2 \bmod 5$, then every layer L_i/L_{i+1} has rank 0 or 6, and therefore $L \cong (\mathbb{Z}_\ell)^6$ for some ℓ dividing m.

On the other hand, if $k \equiv \pm 1 \bmod 5$, then every layer L_i/L_{i+1} has rank 0, 3 or 6. Again the polynomial $t^2 - t - 1$ has two zeroes α and β in $\mathbb{Z}_m$, and if $\lambda \in \{\alpha, \beta\}$, then the elements $x_\lambda = w_1w_4w_5w_6^{-\lambda}$, $y_\lambda = w_2w_4^{-\lambda}w_5w_6$ and $z_\lambda = w_3w_4w_5^{-\lambda}w_6$ generate a G-invariant subgroup of rank 3. It follows that $L \cong (\mathbb{Z}_j)^3 \oplus (\mathbb{Z}_\ell)^3$ for some j and ℓ dividing m, with two possibilities for L for each pair (j, ℓ) such that $j > \ell$: one generated by the images of the (m/j)th powers of x_α, y_α and z_α and the (m/ℓ)th powers of x_β, y_β and z_β, and the other with the roles of α and β reversed.

Similarly, when $k = 5$, every layer L_i/L_{i+1} has rank 0, 3 or 6, but in this case the polynomial $t^2 - t - 1$ has a zero in $\mathbb{Z}_m$ only when $m = k = 5$, and it follows that if some layer L_i/L_{i+1} has rank 3, then the next layer L_{i+1}/L_{i+2} must have rank 6. Hence for $e > 1$, we have $L \cong (\mathbb{Z}_\ell)^6$ or $L \cong (\mathbb{Z}_{5\ell})^3 \oplus (\mathbb{Z}_\ell)^3$ for some ℓ dividing $m/5$. Any subgroup of the latter form is generated by the images of the (m/ℓ)th powers of the elements x_λ, y_λ and z_λ (given above) and the $(m/5\ell)$th powers of all the w_i.

Finally, when $k = 2$, an easy analysis of the situation for the case $m = 2^3 = 8$ shows that $L/K^{(m)}$ can have at most one non-trivial layer of rank less than 6. More specifically, the smallest non-trivial layer (which we might call the 'top' layer) must have rank 4, 5 or 6, and all other non-trivial layers have rank 6. This top layer can be specified in terms of any one of the four non-trivial proper G-invariant subgroups of $K/K^{(2)}$ found above, or have rank 6, and it then determines $L/K^{(m)}$ uniquely.

Putting these observations together, we find that the only possibilities for a normal subgroup L of G contained in K with index $|K : L|$ being a power of a prime k are those included in Table 7.1.

Index $\lvert K:L \rvert$	Generating set for L	Quotient K/L
$d^6 = k^{6t}$, for any k	w_i^d for all i	$(\mathbb{Z}_d)^6$
$c^3d^3 = k^{3(s+t)}$, with $s < t$, for $k \equiv \pm 1 \bmod 5$	$x_\alpha^c,\ y_\alpha^c,\ z_\alpha^c,\ x_\beta^d,\ y_\beta^d,\ z_\beta^d$, or $x_\beta^c,\ y_\beta^c,\ z_\beta^c,\ x_\alpha^d,\ y_\alpha^d,\ z_\alpha^d$	$(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_d)^3$
$125d^6 = 5^{6s+3}$	$x_\lambda^d,\ y_\lambda^d,\ z_\lambda^d$, and all w_i^{5d}	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{5d})^3$
$2d^6 = 2^{6s+1}$	$(w_1w_3)^d,\ (w_2w_3)^d,\ (w_4w_5)^d,\ (w_5w_6)^d$, all w_i^{2d}, and one of w_1^d, w_4^d or $(w_1w_4)^d$	$(\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{2d}$
$4d^6 = 2^{6s+2}$	$(w_1w_3)^d,\ (w_2w_3)^d,\ (w_4w_5)^d,\ (w_5w_6)^d$, and all w_i^{2d}	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{2d})^2$

Table 7.1: Possibilities for $\langle h, a, pq\rangle$-invariant subgroup L of K when $\langle h, a, pq\rangle/K \cong A_5$

Next, we consider which of the $\langle h, a, pq\rangle$-invariant subgroups of $K/K^{(m)}$ are normalized by the additional generator p of the larger group G_3. Here we will work inside the group G_3/N', and adopt the same notation for images of elements in this group.

Note first that if λ is a zero of $t^2 - t - 1$ in $\mathbb{Z}_m$, then so is $1 - \lambda$, and conjugation by p (which lies in G_3/N' but not G_2^1/N') takes

$$\begin{array}{lcl} x_\lambda = w_1w_4w_5w_6^{-\lambda} & \mapsto & w_4^{-1}w_1^{-1}w_3^{-1}w_2^{\lambda} = (x_{1-\lambda})^{-1}(y_{1-\lambda})^{\lambda}(z_{1-\lambda})^{-1}, \\ y_\lambda = w_2w_4^{-\lambda}w_5w_6 & \mapsto & w_6^{-1}w_1^{\lambda}w_3^{-1}w_2^{-1} = (x_{1-\lambda})^{\lambda}(y_{1-\lambda})^{-1}(z_{1-\lambda})^{-1}, \\ z_\lambda = w_3w_4w_5^{-\lambda}w_6 & \mapsto & w_5^{-1}w_1^{-1}w_3^{\lambda}w_2^{-1} = (x_{1-\lambda})^{-1}(y_{1-\lambda})^{-1}(z_{1-\lambda})^{\lambda}. \end{array}$$

Hence in particular, if t^2-t-1 has two distinct zeroes α and β in $\mathbb{Z}_m$, then conjugation by p interchanges the rank 3 subgroups generated by $\{x_\alpha^\ell, y_\alpha^\ell, z_\alpha^\ell\}$ and $\{x_\beta^\ell, y_\beta^\ell, z_\beta^\ell\}$ for each divisor ℓ of m, while if there is just one zero λ (namely in the case $m = k = 5$), then the rank 3 subgroup generated by $\{x_\lambda, y_\lambda, z_\lambda\}$ is preserved.

On the other hand, in the case $k = 2$, we note that conjugation by p interchanges w_1w_3 with w_4w_5, and w_2w_3 with w_5w_6, and w_1 with w_4, and so fixes w_1w_4. Hence p preserves the rank 4 subgroup generated by $S = \{w_1w_3, w_2w_3, w_4w_5, w_5w_6\}$ and the rank 5 subgroup generated by $S \cup \{w_1w_4\}$, but interchanges the rank 5 subgroups generated by $S \cup \{w_1\}$ and $S \cup \{w_4\}$.

It follows that all of the covers of the Petersen graph that arise in the cases described by rows 1, 3 and 5 of Table 7.1, and one of the three cases in row 4, admit a 3-arc-regular group of automorphisms, while the others do not. Also none of these covers can be 4- or 5-arc-transitive, by [12, Theorem 3], and each of the 3-arc-regular covers is unique up to isomorphism, while the 2-arc-regular covers come in pairs.

Thus we have the following theorem:

Theorem 7.1 *Let $m = k^e$ be any power of a prime k, with $e > 0$. Then the arc-transitive regular covers of the Petersen graph with abelian covering group of exponent m are as follows:*

(a) *If* $k \equiv \pm 2$ *mod* 5 *and* $k > 2$*, there is exactly one such cover, namely a* 3*-arc-regular cover with covering group* $(\mathbb{Z}_m)^6$.

(b) *If* $k \equiv \pm 1$ *mod* 5*, then there are exactly* $e+1$ *such covers, namely a* 3*-arc-regular cover with covering group* $(\mathbb{Z}_m)^6$ *plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^3$ *for each proper divisor* ℓ *of* m.

(c) *If* $k = 5$*, then there are exactly two such covers, namely one* 3*-arc-regular cover with covering group* $(\mathbb{Z}_m)^6$ *and one* 3*-arc-regular cover with covering group* $(\mathbb{Z}_{m/5})^3 \oplus (\mathbb{Z}_m)^3$.

(d) *If* $k = 2$*, then there are exactly* 4 *such covers, namely a* 3*-arc-regular cover with covering group* $(\mathbb{Z}_m)^6$*, plus one* 3*-arc-regular cover with covering group* $(\mathbb{Z}_{m/2})^4 \oplus (\mathbb{Z}_m)^2$*, plus one* 2*-arc-regular cover and one* 3*-arc-regular cover with covering group* $(\mathbb{Z}_{m/2})^5 \oplus \mathbb{Z}_m$.

Remark 7.2 *In case* (c) *of the above theorem, taking* $m = 5$ *gives a* 3*-arc-regular cubic graph of order* 1250*, which turns out to have diameter* 10 *and girth* 16*. In fact, at the time of writing this is the largest known connected* 3*-valent graph of diameter* 10*, and its discovery motivated the research behind this paper. It was found almost by accident in* 2006 *by the first author, as a result of a computation to find all symmetric cubic graphs of order up to* 2048 (*which he has since extended to find all symmetric cubic graphs of order up to* 10000).

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