Egon Schulte
Discrete structures in geometry

1 Individual Project's contribution to the CRP

This project will support research collaborations, with various PI's and AP's on our team, on discrete structures in geometry, graph theory and combinatorics, such as polyhedra, polytopes, maps, complexes and tessellations, with symmetry as the unifying theme. A major goal is to explore unexploited connections between these structures.

1.1 Aims and Objectives

A graph-theoretical approach to polyhedra and complexes A main focus is the analysis and classification of highly-symmetric figures in ordinary Euclidean three-space, specifically the radically new *skeletal* approach to polyhedra-like structures (initiated by Grünbaum in 1975). In a nutshell, this skeletal approach breaks away from the traditional notion of polyhedra as solid figures in space that are bounded by faces spanned by membranes, and permits more general discrete polyhedral structures with convex or non-convex, planar or skew, and finite or infinite (helical or zig-zag) polygonal faces and vertex-figures. This approach to polyhedra is essentially graph-theoretical and promises a rich cross-fertilization between areas in discrete geometry and graph theory. Via their skeletal nature as (generally periodic) geometric graphs in space, these figures also have considerable potential for applications in the design and synthesis of molecular structures ("crystal nets") appearing in crystallography and crystal chemistry.

Building on the enumeration of the (rich class of forty-eight) "new" regular polyhedra by Grünbaum and Dress around 1980 (with a simpler approach described in my *Abstract Regular Polytopes* book with McMullen), as well as my recent classification (of the even richer class) of chiral polyhedra, I am planning to continue a joint project with my former PhD student Daniel Pellicer on the enumeration of the regular (that is, flag-transitive) *polygonal complexes* in space. Polygonal complexes are discrete polyhedra-like structures (of rank 3) in space with finite or infinite polygons as 2-faces and with finite graphs as vertex-figures, such that edges may be contained in more than two polygonal faces (not just two, as for ordinary polyhedra). We establish basic structure results for the symmetry groups, discuss geometric and algebraic aspects of operations on their generators, and fully enumerate certain types of simply flag-transitive complexes. We expect to be able to complete the full classification and describe in Part II the details of the geometry and combinatorics of these complexes.

There are several other important classes of highly-symmetric figures whose symmetry groups have transitivity properties weaker than flag-transitivity, and I am planning to investigate them. These classes include: the fully transitive polyhedra; the two-orbit polyhedra (with two flag orbits); the infinite "regular polyhedra of index 2", the uniform polyhedra (the finite polyhedra with planar faces were described in a classical paper by Coxeter, Longuet-Higgins and Miller in 1954). These polyhedral structures are fundamental geometric objects in space, and will stay in mathematics, independent of trends or fashion.

Chiral Polytopes Abstract chirality is a fascinating phenomenon which does not occur in the classical theory of convex polytopes, and hence is more difficult to comprehend. In rank 3, finite examples are given by the irreflexible maps on surfaces. The quest for chiral polytopes of higher ranks has inspired a lot of activity in abstract polytopes. Until very recently, finite chiral polytopes were only known to exist in ranks 3 and 4. However, in an important recent paper by Conder, Hubard and Pisanski, finite chiral polytopes of rank 5 were discovered. This reopened the search for higher rank examples. In a recent joint paper with Antonio Breda and Gareth Jones, we exploit the concept of the chirality group of polytopes and describe a parasite construction for chiral polytopes, which in particular leads to infinite families of finite chiral polytopes of rank 5. Very recently, Daniel Pellicer established the existence of chiral polytopes for any rank. With the existence settled, the focus now is on classification results for chiral polytopes.

My current PhD student Gabriel Cunningham is extensively studying general properties of the chirality group of polytopes and has been able to make significant progress. I hope that some of his findings will advance my long-term goal of completely classifying the universal locally toroidal chiral polytopes of rank 4.

Graphs derived from polytopes, and vice versa Polytope theory can provide important new insights into the structure of highly symmetric graphs, for example, via medial layer graphs. My current PhD student Mark Mixer has investigated various transitivity properties of interesting classes of graphs related to polytopes, including medial layer graphs. Another ongoing project concerns a construction of abstract polytopes from Cayley graphs of symmetric groups. Given any connected graph G with p vertices and q edges, we associate with G a Cayley graph $\mathcal{G}(G)$ of the symmetric group S_p and then construct a vertex-transitive simple abstract polytope of rank q, the *graphicahedron*, whose edge graph (1-skeleton) is $\mathcal{G}(G)$. This generalizes the well-known permutahedron, which is obtained when the graph is a path. The construction yields many interesting highly-symmetric polytopes, including various locally toroidal polytopes. There are various generalizations of this construction and we are planning to explore them in detail.

Polytopes and incidence geometries I am also proposing to continue my work with Dimitri Leemans on links between abstract polytopes and incidence geometries. In particular, in two previous articles we fully described all regular polytopes of rank at least 4, whose automorphism group is a projective linear group $\mathrm{PSL}(2,\mathrm{q})$ ($= \mathrm{L}_2(\mathrm{q})$) or $\mathrm{PGL}(2,\mathrm{q})$. For $\mathrm{PSL}(2,\mathrm{q})$ there are just two regular polytopes, both locally projective regular polytopes of rank 4, namely Grünbaum's 11-cell with group $\mathrm{PSL}(2,11)$ and Coxeter's 57-cell with group $\mathrm{PSL}(2,19)$. The edge graph of the 11-cell is the complete graph K_{11} on 11 vertices. By contrast, there are many chiral 4-polytopes with a group of type $\mathrm{PSL}(2,\mathrm{q})$. Surprisingly, for $\mathrm{PGL}(2,\mathrm{q})$ there is just one regular polytope, the 4-simplex with edge graph K_5.

We are planning to study the polytopes associated with other interesting simple groups or almost simple groups. In particular, we hope to be able to work out the classification for the Ree groups.

1.2 Methodologies

In the research we will use methods of graph theory, geometry and algebra. All research starts with the study of relevant literature, continues with the development of new theories and ends with mathematical proofs. When dealing with polytopes the geometrical aspect becomes very important since most of the proofs are done by construction.

2 Information on funding

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