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Graph representations, configurations and maps

1 Individual Project's contribution to the CRP

1.1 Aims and Objectives

Coherent theory of representations. Motivated by the success of the computer system VEGA [27] that we have been developing since the 90s we also began developing a theory of representations of graphs. In Godsil and Royle [10] a representation r of a graph $G = (V, E)$ is defined as a mapping r from $V(G)$ to some Euclidean space E^n. In Pisanski and Žitnik [30] we presented the foundation of a theory of graph representations with applications to graph drawing. We extended the definition of a graph representation in two ways: $r : V(G) \to M$ is now a mapping from the vertex set to any nonempty set M and is no longer limited to the Euclidean space. On the other hand, for the mapping $r : E(G) \to P(M)$, a mapping from the edge set to the power set $P(M)$, we require that the representation of an edge contains the representation of its end-vertices. This approach allows for further extensions. A representation can now be a mapping from a graph to an incidence structure with a property that vertices are mapped to points and edges are mapped to blocks of the incidence structure with a sole axiom that graph arcs (i.e. incident vertex-edge pairs) are mapped to flags of the incidence structure. Our objective is to further formalize theory of representations and present a coherent and plausible framework for integration of various aspects.

Graph drawing algorithms. By introducing appropriate energy models for representation of graphs one can provide theoretical grounds for various graph drawing methods. In VEGA several such methods were developed like Schlegel diagrams [29] based on the well-known Tutte method for drawing planar graphs [31]. In a similar way, representations based on eigenvalues and eigenvectors of the graph Laplacian (Pisanski, Shawe-Taylor, Fowler [9]) were explored in several papers (e.g. [3]). The study of nodal domains (e.g. the book by Leydold and Stadler [4]) has a variety of applications (Graovac [13]). Many theoretical questions concerning graph representations found practical applications in graph drawing algorithms implemented in the package Pajek [32].

One fundamental question asks how the symmetry of a graph [17] can be transported to the symmetry of graph representation, i.e. how much algebraic symmetry can be represented as a geometric symmetry. For extremely large graphs minimizing the energy of representation presents a computational bottleneck unless the iterations are simplified. This happens when orbits are considered as elements.

An algorithm of Imrich et al. [16] for decomposing graphs into prime factors with respect to the Cartesian products was implemented by Žitnik and built into VEGA. This has enabled us to represent Cartesian products of graphs as a product of representation of its factors. Recent research on near-products of graphs by Imrich et al. will enable us to represent near-products in a way that will expose the product's structure. There are several applications related to near-products. For example, it is natural to represent covering graphs and graph bundles in such a way that they reveal product structure of its vertices. The possibility of choosing the energy of representation and solving the related optimization problems also enabled us to address the question of unit-distance graphs (Horvat et al. [15]).

Our objective is to collect the knowledge from various above mentioned fields, try to achieve new theoretical results and build a program library for newly developed specialized graph drawing methods, which will be used for geometrical representations of various combinatorial objects (configuration, maps, incidence geometries) as well as with large networks.

Representations of Symmetric Configurations. Our work on configurations started around 2000 (Betten, Brinkmann, Pisanski [2]). The key problem of the modern theory of configurations of points and lines is the question which combinatorial configurations have faithful geometric representations (Grünbaum [12]). Our contribution to this problem is the concept of a polycyclic configuration (Boben, Pisanski [5]) in which covering graphs play an essential role. This approach was later extended in a series of papers by Boben, Grünbaum et al. [1, 6, 28]. Note that the problem of geometric representation of combinatorial configurations can be phrased as a question of representation of the associated vertex-colored Levi graphs (Pisanski, Žitnik [26]). Sometimes the connection between some classes of graphs and configurations is quite unexpected. For instance, in Hladnik, Marušič and Pisanski [14] the connections between Haar norm, dihedral groups, cyclic coverings of a dipole and cyclic configurations were presented. We plan to extend these results to more general Haar graphs and generalized dihedral group.

Our objective is to study special classes of configurations where symmetries can be used to achieve "nicer" geometrical and other representations. In particular, the cooperation with other IPs (Marušič, Škoviera) will be required to fully exploit algebraic properties of certain families in geometrical sense.

Representations of Maps and Abstract Polytopes. Recently we extended the study of maps and polyhedra to higher rank abstract polytopes (Conder, Hubard, Pisanski [7]; Monson, Pisanski, Schulte, Weiss [22]) and incidence geometries (cf. Leemans [18, 19]).

The theory of maps and abstract polytopes was first developed for the most symmetric case, i.e. the regular maps and polytopes [21]. Only recently the existence of chiral polytopes of rank 5 was established [7] and later extended to arbitrarily large ranks (Pellicer [24]). By relaxing the symmetry conditions of maps we obtain k-orbit maps [23]. We plan to work on extending the theory of k-orbit maps to polytopes and some other incidence geometries together with relevant operations producing maps in certain classes of k-orbit maps, polytopes and incidence structures with a goal of constructing relevant representations.

We would like to develop an appropriate theory of representations of incidence geometries and abstract polytopes that can be implemented in VEGA. VEGA already contains a set of programs that implement various map operations, e.g. medial, truncation, petrial and dual (Fowler and Pisanski [8]; Pisanski and Randić [25]). We will extend these operations to abstract polytopes and in some extent to other incidence geometries. We would also like to explore various models of radial representations of graphs and maps that are related to the distance and growth in graphs (Luksič [20]; Graves, Pisanski and Watkins [11]).

1.2 Methodologies

In the research we will use methods of combinatorics, graph theory, computer science and algebra. **Coherent theory of representations:** study of literature, developing theory, mathematical proofs. **Graph drawing algorithms:** study of literature, computer programming, testing and improving, proving useful structural properties. **Representations of Symmetric Configurations:** development of algorithms, programming, proving structural theorems, classification of interesting families. **Representations of Maps and Abstract Polytopes:** study of structural properties in literature, identification of interesting families, combining tools in computational algebra with numerical algorithms for geometrical representation.

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